ARTICLE XII.

De Calculo Eclipsium Besseliano Commentatio. Auctore Dr. Gustavo Adolpho Jahn. Lipsiæ, 1848. *Read Feb. 2d*, 1849.

INTRODUCTIO.

Differentiam longitudinum locorum duorum in terra continenti sitorum ex observationibus astronomicis constituere, inter difficillima geographiæ mathematicæ problemata a viris rei gnaris, numeratur; qui quidem ad illud solvendum inter alias observationes potissimum eclipsium et verarum et apparentium usque ad nostros dies adhibuerunt. Negari quidem nequit, calculum differentiæ longitudinum haud difficulter componi ex veris eclipsibus observatis, hanc tamen non accurate exinde erui posse propter penumbram, inter omnes constat. Multo accuratior calculus ope apparentium eclipsium constitui potest, quum apparentes subtilius et politius observari quam veras natura rei patiatur: altera vero ex parte rationes, quæ exinde subducuntur, difficillimæ et nimis vagæ sunt, propterea quod parallaxes ibi occurrunt, cujus quantitatis formulas in summam, quæ fieri potest, simplicitatem redactas nondum videmus, licet multi summam huic rei navaverint operam. Ad hanc calculi difficultatem id accedit, quod, quantum Telluris forma a figura globi differat, ante nostra tempora non satis constabat. Quibus igitur rebus factum est, ut olim ad computandam differentiam, quæ sit inter geographicas duorum locorum longitudines, sæpius veræ adhiberentur, quam apparentes eclipses. Quæ autem Telluris compressio quum hoc sæculo in nostram rem accuratius definita esse videatur, item tubi validiores, calculandi methodi accuratius et in usum commodius compositæ, atque catalogi fixarum ephemeridesque ad altiorem sint adductæ perfectionem: observationes eclipsium apparentium præstantia superare observationes verarum, astronomi non fateri non potuerunt.

Ex ejusmodi autem eclipsium natura duplex, isque diversus rationum subducendarum modus oritur, cujus vetustiorem, qua semper sequaris singularum observationis partium seriem, propter vim parallaxium longinquitate laborare necesse est. Ad quam longinquitatem tollendam astronomis, qui his ultimis quinquaginta annis floruerunt, hoc curæ cordique erat, ut formula invenirent, quibus facillimo modo parallaxium vis exprimeretur.

Altera methodo adhibita id tantum agitur, ut, neglecta ordine singulorum momentorum

eclipsis apparentis, quæratur, quæ relatio tempore geocentricæ conjunctionis intercedat inter positiones veras atque semidiametros utriusque corporis cœlestis. Quarum vero ad rem quum nec computatione positionum apparentium utriusque corporis opus sit, nec quantitate compressionis Telluris accuratissime definita: rationes majori cum facilitate et celeritate subduci posse facile patet. Quare id miraberis, quod vestigia celleberrimi Lagrange, qui illius solutionis auctor est, nemo hucusque presserit, nec in usum differentiæ longitudinum eruendæ, ut saltem equidem comperi, hoc problema verterit. Egregiam enim Clausenii* solutionem, in qua de tempore minimæ distantiæ geometricæ, ut quantitate ignota, quæritur, ad nostram rem ægre poteris transferre. Besselius demum, Lagrangium secutus, elegantissimam et in usum calculi facillimam hujus problematis solutionem ante oculos posuit in Astronomische Nachrichten von Schumacher No. 151 u. 152, unter dem titel: Beiträge zur Theorie der Finsternisse und den Berechnungs-Methoden derselben. Quæ autem res in disputatione ab Besselio tractata geographiæ mathematicæ tantum utilitatis affert, ut non possimus non vehementer optare, ut formulæ ibi exquisitæ ad constituendam longitudinum differentiam adhibeantur: atqui illam rem ii tantum omni ex parte perspicere et intelligere possunt, qui in definienda longitudinum differentia multum et fere semper versantur. Quare operæ pretium nobis visum est, æquationes et formulas eadem qua Besselius serie accuratius in hocce libello explicare, annotationibus exornare atque exempla quædam in calculi usum communem, in quem Besselius nihil adjecit, addere.

PARS PRIMA.

De argumentatione analytica æquationum expressionum que a Besselio compositarum.

§ 1. Besselius, commemorata illa quam alio loco† docuerat, præ computatione occultationum fixarum, methodos disquirit, quæ in introductione a me jam expositæ sunt atque rationibus subducendis ex observatis solis et fixarum eclipsibus inserviunt. Quarum methodorum priorem, si respiciatur distantia longitudinum duorum observatoriorum, nondum accurate explicatam esse in secunda sectione docet, viam expediens, qua conjunctionis veræ ejusque correctæ tempus possit inveniri. Deinde ad alterum sese confert methodum, ita ut demonstret, methodum priorem ab ea superari non solum eo, quod positiones utriusque corporis cœlestis apparentes evitentur, sed eo etiam, quod faciliori computandi ratione multo celerius res perducatur ad finem. Quæ celeritas ita efficitur, ut, non tempus geocentricæ conjunctionis, sed ipsa observatorii longitudo geographica sumatur quantitas quæsita. Tertia porro in sectione ad solutionem perficiendam signa quantitatum constituit, quæ, etsi rei non satis convenire nobis videantur, hoc loco reservaturi et breviter descripturi sumus.

Litteræ α, δ, ϱ, π significent propioris corporis cœlestis argumenta vera, scilicet rectascensionem, declinationem, horizontalem semidiametrum et parallaxem æquatoralem; α',

* Astron. Nachrichten von Schumacher, No. 40.

† Astron. Nachr., No. 145. unter dem Titel: Ueber die Vorausberechnung der Sternbedeckungen. Von Herrn Professor Bessel.

δ', ϱ', apparentia earum argumenta; in remotioris corporis usum vero A, D, R, π' vera argumenta, nimirum rectascensionem, declinationem, horizontalem semidiametrum et parallaxem æquatoralem; sed A', D', R' earum argumenta apparentia; littera μ porro significetur tempus observationis in gradus redactum, id est siderale; φ' geocentrica elevatio poli, atque r distantia, quæ inter centrum Telluris et observatoris positionem interposita est; denique demidiam Telluris axem majorem et minorem significent $\mathfrak{A}$, $\mathfrak{B}$; distantiam longitudinum ex oriente positive sumtam d, excentricitatis quadratum e^2, compressionem Telluris χ, distantiasque, quæ intercedunt inter corpus cœleste aut propius aut remotius atque Telluris centrum, ϑ et ϑ'. Quibus enarratis, expedire nobis liceat viam, facillime inveniendi systema æquationum [r]:

$$\begin{aligned}
\triangle \cos\delta' \sin\alpha' &= \cos\delta \sin\alpha - r\cos\varphi' \sin\pi \sin\mu \\
\triangle \cos\delta' \cos\alpha' &= \cos\delta \cos\alpha - r\cos\varphi' \sin\pi \cos\mu \\
\triangle \sin\delta' &= \sin\delta - r\sin\varphi' \sin\pi \\
\triangle' \cos D' \sin A' &= \cos D \sin A - r\cos\varphi' \sin\pi' \sin\mu \\
\triangle' \cos D' \cos A' &= \cos D \cos A - r\cos\varphi' \sin\pi' \cos\mu \\
\triangle' \sin D' &= \sin D - r\sin\varphi' \sin\pi'.
\end{aligned}$$

§ 2. Si igitur Telluris centrum A (Fig. 1) habetur origo coordinatarum linearum systematis rectiangularis, in quo Y A X æquatoris aream denotet, et A X lineam æquinoctiorum, M vero observatorium in Telluris superficie situm, M′ corpus cœleste propius et M″ remotius; erit

$$\begin{array}{lll}
\angle QAP=\mu & \angle Q'AP'=\alpha & \angle Q''AP''=A \\
\angle MAP=\varphi' & \angle M'AP'=\delta & \angle M''AP''=D \\
MA=r & M'A=\vartheta & \angle M''A=\vartheta'
\end{array}$$

ideoque, *si* x, y, z; x', y', z'; x'' y'' z'' sunt signa coordinatarum punctorum M, M′ et M″:

$$\left.\begin{array}{l|l|l}
x=r\cos\mu\cos\varphi' & x'=\vartheta\cos\alpha\cos\delta & x''=\vartheta'\cos A\cos D \\
y=r\sin\mu\cos\varphi' & y'=\vartheta\sin\alpha\cos\delta & y''=\vartheta'\sin A\cos D \\
z=r\sin\varphi' & z'=\vartheta\sin\delta & z''=\vartheta'\sin D
\end{array}\right\} \quad (1.)$$

Fingas tibi M originem alius coordinatarum systematis ejusque rectiangularis, cujus coordinatæ punctorum M et M″, significentur litteris ξ', η', ζ'; ξ'', η'', ζ''; tunc habebis formulas æquationibus in (1.) analogas:

$$\left.\begin{array}{ll}
\xi'=\vartheta_{\prime}\cos\alpha'\cos\delta' & \xi''=\vartheta'_{\prime}\cos A'\cos D' \\
\eta'=\vartheta_{\prime}\sin\alpha'\cos\delta' & \eta''=\vartheta'_{\prime}\sin A'\cos D' \\
\zeta'=\vartheta_{\prime}\sin\delta' & \zeta''=\vartheta'_{\prime}\sin D'
\end{array}\right\} \quad (1.\ a.)$$

in quibus $\vartheta_{\prime}$=MM′ atque $\vartheta'_{\prime}$=MM″. Sed iterum

$$\left.\begin{array}{ll}
\xi'=x'-x & \xi''=x''-x \\
\eta'=y'-y & \eta''=y''-y \\
\zeta'=z'-z & \zeta''=z''-z
\end{array}\right\} \quad (1.\ b.)$$

esse constat. Ex systematibus igitur (1. a.) et (1. b.) compositis derivatur:

$$\begin{array}{ll}
\vartheta_{,} \cos \alpha' \cos \delta' = x' - x & \vartheta_{,}' \cos \mathfrak{A}' \cos D' = x'' - x \\
\vartheta_{,} \sin \alpha' \cos \delta' = y' - y & \vartheta_{,}' \sin \mathfrak{A}' \cos D' = y'' - y \\
\vartheta_{,} \sin \delta' = z' - z & \vartheta_{,}' \sin D' = z'' - z
\end{array}$$

Si vero ex (1) valores quantitatum, quæ ex dextra parte signi æqualitatis in his æquationibus positæ sunt, substitueris, facile patet esse

$$\begin{array}{lll}
\vartheta_{,} \cos \alpha' \cos \delta' & = \vartheta \cos \alpha \cos \delta & - r \cos \mu \cos \varphi' \\
\vartheta_{,} \sin \alpha' \cos \delta' & = \vartheta \sin \alpha \cos \delta & - r \sin \mu \cos \varphi' \\
\vartheta_{,} \sin \delta' & = \vartheta \sin \delta & - r \sin \varphi' \\
\vartheta_{,}' \cos \mathfrak{A}' \cos D' & = \vartheta' \cos \mathfrak{A} \cos D & - r \cos \mu \cos \varphi' \\
\vartheta_{,}' \sin \mathfrak{A}' \cos D' & = \vartheta' \sin \mathfrak{A} \cos D & - r \sin \mu \cos \varphi' \\
\vartheta_{,}' \sin D' & = \vartheta' \sin D & - r \sin \varphi'.
\end{array}$$

Et si harum æquationem priores tres per $\vartheta_{,}$ posteriores vero per ϑ' diviseris, littera Δ pro $\frac{\vartheta_{,}}{\vartheta}$ et Δ' pro $\frac{\vartheta_{,}'}{\vartheta'}$ posita,* has invenies æquationes:

$$\left.\begin{array}{lll}
\Delta \cos \alpha' \cos \delta' & = \cos \alpha \cos \delta & - \frac{r}{\vartheta} \cos \mu \cos \varphi' \\
\Delta \sin \alpha' \cos \delta' & = \sin \alpha \cos \delta & - \frac{r}{\vartheta} \sin \mu \cos \varphi' \\
\Delta \sin \delta' & = \sin \delta & - \frac{r}{\vartheta} \sin \varphi' \\
\Delta' \cos \mathfrak{A}' \cos D' & = \cos \mathfrak{A} \cos D & - \frac{r}{\vartheta'} \cos \mu \cos \varphi' \\
\Delta' \sin \mathfrak{A}' \cos D' & = \sin \mathfrak{A} \cos D & - \frac{r}{\vartheta'} \sin \mu \cos \varphi' \\
\Delta' \sin D' & = \sin D & - \frac{r}{\vartheta'} \sin \varphi'
\end{array}\right\} \text{(1. c.)}$$

Est porro $\frac{r}{\vartheta} = \frac{r}{A} \cdot \frac{A}{\vartheta}, \frac{r}{\vartheta'} = \frac{r}{A} \cdot \frac{A}{\vartheta'}$, atque $\frac{r}{A} = 1$, si $A = 1$, igitur $\frac{A}{\vartheta} = \sin \pi, \frac{A}{\vartheta'} = \sin \pi'$, atque $\frac{r}{\vartheta} = r \sin \pi, \frac{r}{\vartheta'} = r \sin \pi'$.

Quibus valoribus quotientium $\frac{r}{\vartheta}, \frac{r}{\vartheta'}$ substitutis, systematis (1. c.) facies hæc est:

$$\left.\begin{array}{lll}
\Delta \cos \delta' \sin \alpha' & = \cos \delta \sin \alpha & - r \cos \varphi' \sin \mu \sin \pi \\
\Delta \cos \delta' \cos \alpha' & = \cos \delta \cos \alpha & - r \cos \varphi' \cos \mu \sin \pi \\
\Delta \sin \delta' & = \sin \delta & - r \sin \varphi' \sin \pi \\
\Delta' \cos D' \sin \mathfrak{A}' & = \cos D \sin \mathfrak{A} & - r \cos \varphi' \sin \mu \sin \pi' \\
\Delta' \cos D' \cos \mathfrak{A}' & = \cos D \cos \mathfrak{A} & - r \cos \varphi' \cos \mu \sin \pi' \\
\Delta' \sin D' & = \sin D & - r \sin \varphi' \sin \pi'
\end{array}\right\} \text{(I.)}$$

quo in systemate insunt æquationes quæ sitæ.

§ 3. Brevitatis causa, quod ipse Besselius fecit, proponatur:

$$\begin{array}{ll}
\cos \delta \sin \alpha - r \cos \varphi' \sin \mu \sin \pi & = a \\
\cos \delta \cos \alpha - r \cos \varphi' \cos \mu \sin \pi & = b \\
\sin \delta \quad - r \sin \varphi' \sin \pi & = c
\end{array}$$

* Quare Besselii sententia non plane vera est, ex qua signa Δ, Δ' distantias, quæ sint inter utrumque corpus cœleste et terræ centrum denotent.

$$\begin{aligned}\cos D \sin \mathcal{A} - r \cos \varphi' \sin \mu \sin \pi' &= a' \\ \cos D \cos \mathcal{A} - r \cos \varphi' \cos \mu \sin \pi' &= b' \\ \sin D \quad - r \sin \varphi' \sin \pi' &= c'\end{aligned}$$

unde sequitur, ratione formularum (I) habita:

$$\begin{aligned}\Delta \cos \delta' \sin \alpha' &= a \qquad & \Delta' \cos D' \sin \mathcal{A}' &= a' \\ \Delta \cos \delta' \cos \alpha' &= b & \Delta' \cos D' \cos \mathcal{A}' &= b' \\ \Delta \sin \delta' &= c & \Delta' \sin D' &= c'\end{aligned}$$

igitur

$$\begin{aligned}\Delta\Delta' \cos \delta' \sin \alpha' \cos D' \sin \mathcal{A}' &= aa' \\ \Delta\Delta' \cos \delta' \cos \alpha' \cos D' \cos \mathcal{A}' &= bb' \\ \Delta\Delta' \sin \delta' \sin D' &= cc',\end{aligned}$$

quibus ex tribus æquationibus additis petitur

$$\Delta\Delta' \cos \delta' \cos D' (\cos \alpha' \cos \mathcal{A}' + \sin \alpha' \sin \mathcal{A}') + \Delta\Delta' \sin \delta' \sin D' = aa' + bb' + cc',$$

id est

$$\Delta\Delta' \left\{ \cos \delta' \cos D' \cos (\alpha' - \mathcal{A}') + \sin \delta' \sin D' \right\} = aa' + bb' + cc'. \ldots (2.)$$

Factor autem, cum factore $\Delta\Delta'$ conjunctus, æquiparat cosinum apparentis distantiæ quæ inter utriusque corporis centra est; si enim in triangulo M′ P M″ (Fig. 2.) M′ et M″ loca apparentia centrorum utriusque corporis cœlestis denotant, atque P significat polum æquatoris, primum invenies:

$$M'P = 90° - \delta', \ M''P = 90° - D', \ < M'PM'' = \alpha' - \mathcal{A}';$$

exinde sequitur:

$$\cos M'M'' = \sin \delta' \sin D' + \cos \delta' \cos D' \cos (\alpha' - \mathcal{A}');$$

atque littera Σ pro M′ M″ posita, æquatio ad (2.) hanc assumit formam:

$$\Delta\Delta' \cos \Sigma = aa' + bb' + cc' \ldots (2.\ a.)$$

Primo autem et ultimo defectus tempore quum sit $\Sigma = \varrho' \pm R'$, ita ut signum + ad externum pertineat tactum, signum — vero ad internum, sequitur

$$\Delta\Delta' \cos \Sigma = \Delta\Delta' \cos (\rho' \pm R')$$
$$\text{i. e. } \Delta\Delta' \cos \Sigma = \Delta\Delta' \cos \rho' \cos R' \mp \Delta\Delta' \sin \rho' \sin R' \ldots (2.\ a.^*)$$

Quum porro secundum (Fig. 1.)

$$\vartheta_{,} : \vartheta = \sin \rho : \sin \rho'$$
$$\vartheta'_{,} : \vartheta' = \sin R : \sin R',$$

ergo

$$\frac{\vartheta_{,}}{\vartheta} \sin \rho' = \sin \rho \qquad \frac{\vartheta'_{,}}{\vartheta'} \sin R' = \sin R \ldots (2.\ b.).$$

Porro ex æquationibus (I.) derivatur

$$\Delta^2 = a^2 + b^2 + c^2 \qquad \Delta'^2 = a'^2 + b'^2 + c'^2 \ldots (2.\ c.)$$

Igitur loco sin ϱ', sin R' in (2. b.) si posueris cosinus, atque in æquationibus inde exortis

$$\Delta^2 \cos \rho'^2 = \Delta^2 - \sin \rho^2 \quad , \quad \Delta'^2 \cos R'^2 = \Delta'^2 - \sin R^2 \ldots (2.\ d.)$$

pro quantitatibus Δ^2, Δ'^2 ex dextra signi æquationis jacentibus substitueris earum valores ex (2. c.), invenies:

$$\left.\begin{array}{l}\Delta \cos \rho' = \sqrt{a^2+b^2+c^2-\sin \rho^2} \\ \Delta' \cos R' = \sqrt{a'^2+b'^2+c'^2-\sin R^2}\end{array}\right\} \ldots (2.\ e.)$$

atque si valores $\Delta \cos \varrho'$, $\Delta' \cos R'$ in (2. e.), et $\Delta \sin \varrho'$, $\Delta' \sin R'$ in (2. b.) definiti, in æquationem (2. a*.) substituuntur, formulam denique invenies:

$$\Delta\Delta' \cos \Sigma = \sqrt{(a^2+b^2+c^2-\sin \rho^2)}.\ \sqrt{(a'^2+b'^2+c'^2-\sin R^2)} \mp \sin \rho \sin R,$$

quam etiam Besselius proposuit.—Si porro hanc formæ $\Delta\Delta' \cos \Sigma$ valorem cum ejusdem formæ valore in (2. a.) invento conjunxeris, fit

$$\sqrt{(a^2+b^2+c^2-\sin \rho^2)}.\ \sqrt{(a'^2+b'^2+c'^2-\sin R^2)} \mp \sin \rho \sin R = aa'+bb'+cc',$$

unde, eliminatis, radicum signis, hanc æquationem habebis

$$\begin{array}{l}(aa'+bb'+cc')^2 \pm 2\,(aa'+bb'+cc') \sin \rho \sin R = \\ (a'^2+b'^2+c'^2)(a^2+b^2+c^2)-(a'^2+b'^2+c'^2) \sin \rho^2-(a^2+b^2+c^2) \sin R^2,\end{array}$$

cujus post facilem transformationem videbis formam hancce:

$$\begin{array}{l}(ab'-a'b)^2+(ac'-a'c)^2+(bc'-b'c)^2 = \\ (a' \sin \rho \pm a \sin R)^2+(b' \sin \rho \pm b \sin R)^2+(c' \sin \rho \pm c \sin R)^2 \ldots (\mathrm{II.})\end{array}$$

Cujus quidem æquationis (II.) deductio, a Besselio (apud quem est [2.]) proposita, longe brevior est, quam cujus ratio facile possit intelligo. Profecto enim, ut unum tantummodo afferamus exemplum, cur ponat formulas tanquam notas

$$\begin{array}{ll}\Delta \sin \rho' = \sin \rho & \Delta \cos \rho' = \sqrt{a^2+b^2+c^2-\sin \rho^2} \\ \Delta' \sin R' = \sin R & \Delta' \cos R' = \sqrt{a'^2+b'^2+c'^2-\sin R^2}.\end{array}$$

Quibus rebus cognitis facile apparet, in æquatione (II.), quamvis observationis tempus t ipsum directe in ea non sit expressum, analysin eclipsium quam maxime universalem in esse atque ea de causa præferendam esse, quod, quum, argumentis apparentibus evitatis, non adhibeantur nisi vera, elementa calculi quæcunque ad Telluris centrum relata sunt. Quam æquationem, quum poli et originis angulorum ad polum jacentium positio ex arbitrio definiri possit, in innumeras alias æquationes posse transformari, Besselius quidem commemoravit, secutus autem est per totam suam disquisitionem æquationis (II.) transformationem:

$$\begin{array}{rl}e^2+f^2+g^2 = & (e \cos u+f \sin u \cos v-g \sin u \sin v)^2 \\ & +(e \sin u-f \cos u \cos v+g \cos u \sin v)^2 \\ & +(f \sin v+g \cos v)^2 \ldots\ldots\ldots\ldots\ (3.)\end{array}$$

cujus identicæ expressionis quantitates u, v angulorum argumenta, pro arbitrio accepta, significant.

§ 4. Æquationem illam, transformandam in posterum differens, ad casum, quem poteris in eclipsibus reperire, simplicissimum, in quarta sectione noster sese confert. Quæ summa simplicitas occurrit, si π' et R', igitur et $R=0$, sive, quod ad idem recurrit, si

fixæ occultatio in calculum vocanda est. Summam vero tantum hujus calculi, quæ his positis ex æquatione (II.) derivatur, quum Besselius commemoraverit, ratione eam inveniendi omissa cocta recoquere nobis non videmur transformationem accuratius propondituri. Posito igitur $\pi'=0$, $R=0$, æquatio (II.) hanc induit formam:

$$(a'^2+b'^2+c'^2)\sin\rho^2=(ab'-a'b)^2+(ac'-a'c)^2+(bc'-b'c)^2\ldots(3.\text{a.})$$

porro formulæ § 3. ineunte exhibitæ abeunt in

$$\begin{aligned} a&=\cos\delta\sin\alpha-r\cos\varphi'\sin\mu\sin\pi & a'&=\cos D\sin A\\ b&=\cos\delta\cos\alpha-r\cos\varphi'\cos\mu\sin\pi & b'&=\cos D\cos A\\ c&=\sin\delta-r\sin\varphi'\sin\pi & c'&=\sin D \end{aligned}$$

ergo $a'^2+b'^2+c'^2=1$. Porro erat in (3.):

$$ab'-a'b=e,\ ac'-a'c=f,\ bc'-b'c=g\ldots(3.\text{a}^*.)$$

Igitur expressio (3. a.), argumentis D et A pro u et v substitutis, abit in

$$\begin{aligned}\sin\rho^2&=(e\cos D+f\sin D\cos A-g\sin D\sin A)^2\\&+(e\sin D-f\cos D\cos A+g\cos D\sin A)^2\\&+(f\sin A+g\cos A)^2.\end{aligned}$$

Jam autem si pro a', b', c' valores modo inventos substitueris in (3. a.*) nempe:

$$\begin{aligned}e&=a\cos D\cos A-b\cos D\sin A\\f&=a\sin D-c\cos D\sin A\\g&=b\sin D-c\cos D\cos A,\end{aligned}$$

has videbis æquationes:

$$\begin{aligned}&e\cos D+f\sin D\cos A-g\sin D\sin A=\cos\delta\sin(\alpha-A)-r\cos\varphi'\sin\pi\sin(\mu-A)\\&e\sin D-f\cos D\cos A+g\cos D\sin A=0\\&f\sin A+g\cos A=-\sin\delta\cos D+\cos\delta\sin D\cos(\alpha-A)+r\sin\pi\{\sin\varphi'\cos D-\cos\varphi'\sin D\cos(\mu-A)\};\end{aligned}$$

et exinde formulam enucleatam:

$$\begin{aligned}\sin\rho^2&=\{\cos\delta\sin(\alpha-A)-r\cos\varphi'\sin\pi\sin(\mu-A)\}^2\\&+\{\sin\delta\cos D-\cos\delta\sin D\cos(\alpha-A)-r\sin\pi[\sin\varphi'\cos D-\cos\varphi'\sin D\cos(\mu-A)]\}^2\end{aligned}$$

Sed ad diametrum ϱ eliminandum proportio $A:\varrho=\sin\varrho:\sin\pi$ adhibetur, ex qua $\sin\varrho=\frac{A}{\varrho}\times\sin\pi$ sequitur, i. e. si $\frac{A}{\varrho}=k$ brevitatis causa ponitur, $\sin\varrho=k\sin\pi$. Si igitur $k\sin\pi$ pro $\sin\varrho$ substitueris, hancce habebis æquationem:

$$\begin{aligned}k^2&=\left\{\frac{\cos\delta\sin(\alpha-A)}{\sin\pi}-r\cos\varphi'\sin(\mu-A)\right\}^2\\&+\left\{\frac{\sin\delta\cos D-\cos\delta\sin D\cos(\alpha-A)}{\sin\pi}-r[\sin\varphi'\cos D-\cos\varphi'\sin D\cos(\mu-A)]\right\}^2\ (\text{III.})\end{aligned}$$

quæ apud Besselium est [3.].

Annotatio. Ex Burckhardtii tabulis est

$$k=0.2725 \qquad \log k=9.4353665.$$

Si vero, brevitatis causa, hisce signis uteris

$$\left.\begin{array}{l} \dfrac{\cos\delta\sin(\alpha-A)}{\sin\pi}=P \\ \dfrac{\sin\delta\cos D-\cos\delta\sin D\cos(\alpha-A)}{\sin\pi}=Q \\ r\cos\varphi'\sin(\mu-A)=u \\ r\sin\varphi'\cos D-r\cos\varphi'\sin D\cos(\mu-A)=v \end{array}\right\}\quad(4.)$$

expressio [III.] in simplicissimam abit hanc formam:

$$k^2=(P-u)^2+(Q-v)^2\ldots(4.\ \mathrm{a.})$$

§ 5. Quantitates k, $P-u$, $Q-v$ aut augendo aut minuendo mutentur necesse est, quia ab elementis α, δ, π, e^2 et k pendent, quæ ipsa observationibus non omni ex parte respondent. Mutationes autem valorum $P-u$ et $Q-v$, secundum Taylorii theorema, per partialia differentialium quota, ratione α, δ, π, et e^2 habita, atque per correctiones $\Delta\alpha$, $\Delta\delta$, $\Delta\pi$, et Δe^2 argumentorum α, δ, π et e^2 exprimi possunt.

Quodsi Z et Z' sunt mutationes valorum $P-u$ et $Q-v$, æquatio (4. a.) abit in hanc

$$(k+\Delta k)^2=(P-u+Z)^2+(Q-v+Z')^2\ldots.(4.\ \mathrm{a.}^*).$$

Sed quum correctiones $\Delta\alpha$, $\Delta\delta$, $\Delta\pi$ et Δe^2, propter summam recentium tabularum astronomicarum perfectionem, minimæ esse quantitatis per se pateat, earum producta et quadrata poteris missa facere, et habebis:

$$Z=a.\ \Delta\alpha+b.\ \Delta\delta+c.\ \Delta\pi+d.\ \Delta e^2$$
$$Z'=a'.\ \Delta\alpha+b'.\ \Delta\delta+c'.\ \Delta\pi+d'.\ \Delta e^2,$$

et hac de re pro (4. a.*)

$$\begin{aligned}(k+\Delta k)^2=&(P-u+a.\ \Delta\alpha+b.\ \Delta\delta+c.\ \Delta\pi+d.\ \Delta e^2)^2\\&+(Q-v+a'.\ \Delta\alpha+b'.\ \Delta\delta+c'.\ \Delta\pi+d'.\ \Delta e^2)^2\ldots(\mathrm{IV.})\end{aligned}$$

qua in æquatione a, b, c, a', b', c' partialia differentialium quota, si respexeris argumenta α, δ, π et e^2, significare supra monitum est. Hæc æquatio Besselio est [4.]. Signa autem a, b, c, a', b', c' non satis commode ab auctore electa mihi videntur, propterea quod in usum aliorum argumentorum, quamvis ab hisce longe diversorum, supra sunt adhibita.

§ 6. His computatis et constitutis, id nunc agendum esse, ut ex æquatione (IV.) tempus primi meridiani, quod æquiparat tempus observationis, i. e. $t-d$, definiatur, sub finem quartæ sectionis dicit. Quam ob rem id in quinta sectione efficere studet, ut argumenta t et d in æquationem (IV.) introducat æquationemque in formam ad computandum faciliorem redigat. Quomodo hac in re progrediatur statim accuratius exposituri sumus.— Significetur per $T+T'$ primi meridiani tempus, ad quod quantitates α, δ, π referuntur; per p et q, tempore T, significentur valores P et Q; eorumque mutationes, intra tempus T', per p' et q'; et denique per P et Q valores P et Q, tempore $T+T'$; his positis sequitur:

$$P=p+p'.\ T'\quad,\quad Q=q+q'.\ T'\ldots(5.).$$

Si pro correctionibus $\Delta\alpha$, $\Delta\delta$, $\Delta\pi$ et Δe^2 substituuntur hæc duæ i et i', locum habeant hæ formulæ:

$$p'.\,i - q'.\,i' = Z \quad , \quad q'.\,i + p'.\,i' = Z' \qquad (5.^*),$$

quarum pro posteriore Besselius hanc

$$q'.\,i + q'.\,i' = a'.\,\Delta\alpha + b'.\,\Delta\delta + c'.\,\Delta\pi + d'.\,\Delta e^2$$

falsam habet. Ceterum quatuor correctiones i, i', i'' et i''' pro his duabus i et i' exspectaveris, si $\Delta\alpha$, $\Delta\delta$, $\Delta\pi$ et Δe^2 eliminandæ sunt. Quum autem $\Delta\pi$ et Δe^2 fere semper sint minutissimæ quantitates, licitum est pro Z et Z' formas $p'.\,i - q'.\,i'$ et $q'.\,i + p'.\,i'$ ponere, id quod ad calculum numericum instituendum esse aptissimum, paulo post demonstraturi sumus.

Quibus igitur valoribus P, Q, Z et Z' in (5.) et (5.*) dictis si in æquatione (IV.) uteris formula oritur

$$(k + \Delta k)^2 = (p + p'.\,T' - u + p'.\,i - q'.\,i')^2 + (q + q'.\,T' - v + q'.\,i + p'.\,i')^2 \ldots (5.\ a.),$$

quæ, si ponitur

$$\left.\begin{array}{ll} p - u = m \sin M & p' = n \sin N \\ q - v = m \cos M & q' = n \cos N \end{array}\right\} \ldots (5.\ b.)$$

abit in hanc æquationem

$$\begin{aligned}(k + \Delta k^2) &= \{m \sin M + (T' + i)\, n \sin N - i'.\, n \cos N\}^2 \\ &\quad + \{m \cos M + (T' + i)\, n \cos N + i'.\, n \sin N\}^2 \\ &= m^2 + n^2 (T' + i)^2 + n^2 i'^2 + 2\, m\, n\, (T' + i) \cos (M - N) - 2\, m\, n\, i' \sin (M - N),\end{aligned}$$

id est

$$(k + \Delta k)^2 = \{m \cos (M - N) + n\, (T' + i)\}^2 + \{m \sin (M - N) - n\, i'\}^2,$$

quæ etiam in Besselii dissertatione legitur. Exinde vero sequitur, quum Δk^2 et i'^2, propter minimam quantitatem, omiseris:

$$\begin{aligned}\pm \sqrt{\{k^2 + 2\, k.\,\Delta k - m^2 \sin (M - N)^2 + 2\, m\, n\, i' \sin (M - N)\}} - m \cos (M - N), \\ = n\, (T' + i).\end{aligned}$$

Si ponitur

$$k = \frac{m \sin (M - N)}{\cos \psi} \ldots\ldots (5\ c.)$$

et radix ex duabus prioribus classibus extrahitur (duæ nimirum illæ classes sufficiunt), erit:

$$T' = -\frac{m \cos (M - N \pm \psi)}{n \cos \psi} - i \pm \frac{i'}{\tan \psi} \mp \frac{\Delta k}{n \sin \psi)}$$

et, quum ψ propter $\cos \psi$ negativos quoque valores habere possit,

$$T' = -\frac{m \cos (M - N \mp \psi)}{n \cos \psi} - i \pm \frac{i'}{\tan \psi} \mp \frac{\Delta k}{n \sin \psi}$$

Sed aptius est, inferiore signo sublato, regulam a Besselio traditam sequi. Existit igitur formula, quæ apud Besselium [5.] est:

$$T' = -\frac{m \cos (M - N - \psi)}{n \cos \psi} - i - \frac{i'}{\tan \psi} - \frac{\Delta k}{n \sin \psi} \ldots\ldots (V.)$$

Ex eo, quod initio hujus §. $T + T' = t - d$ positum erat, sequitur $d = t - T - T'$. Si igitur substituitur valor T' modo inventus, videbimus æquationem, ex qua summa deducitur,

$$d = t - T + \frac{m \cos (M - N - \psi)}{n \cos \psi} + i + \frac{i'}{\tan \psi} + \frac{\Delta k}{n \sin \psi} \dots \text{(VI.)}$$

et quæ apud nostrum [6.] est.

§ 7. In sexta, septima atque octava sectione Besselius plura commemoravit, quæ iis, qui solutione illa usuri sunt, respicienda quidem, sed plurimis intellectu difficiliora sunt. Itaque ea, quæ hac de re accuratius sunt dicenda, quemadmodum in introductione promisimus, in practica demum parte proferemus, atque sufficiet, nostræ rei deductionis tantummodo formularum adjicere, earum, quæ Besselio sunt [7.], [8.], [9.] et [10.].

§ 8. In sexta sectione hoc schema propositum est:

$$\begin{array}{l|lllll}
T - 2^{\mathrm{h}} & a_{,,} & & & & \\
 & & b_{,,} & & & \\
T - 1^{\mathrm{h}} & a_{,} & & c_{,} & & \\
 & & b_{,} & & d_{,} & \\
T^{\mathrm{h}} & a & & c & & e \\
 & & b' & & d' & \\
T + 1^{\mathrm{h}} & a' & & c' & & \\
 & & b'' & & & \\
T + 2^{\mathrm{h}} & a'' & & & &
\end{array}$$

Jam, si ab $a_{,,}$ profectus fueris, formula interpolationis nota habetur.

$$y = a_{,,} + \frac{x}{1} \cdot b_{,,} + \frac{x(x-1)}{1.\,2} c_{,} + \frac{x(x-1)(x-2)}{1.\,2.\,3} d_{,} + \frac{x(x-1)(x-2)(x-3)}{1.\,2.\,3.\,4} e + \dots.$$

Ex hac interpolationis formula ut aliam invenias, qua, si ex a exordiendum est, uti possis, nulla alia re opus erit, quam ut $2 + T'$ substituatur pro x; qua ex substitutione sequitur:

$$\begin{aligned} y = {} & (a_{,,} + 2 b_{,,} + c_{,}) + (b_{,,} + \tfrac{3}{2} c_{,} + \tfrac{1}{3} d_{,} - \tfrac{1}{12} e)\, T' \\ & + (\tfrac{1}{2} c_{,} + \tfrac{1}{2} d_{,} - \tfrac{1}{24} e)\, T'^2 + (\tfrac{1}{6} d_{,} + \tfrac{1}{12} e)\, T'^3 + \tfrac{1}{24} e .\, T'^4 + \dots. \end{aligned}$$

Illud autem schema idem est, quam sequens:

$$\begin{array}{l|l|l|l|l}
a_{,,} & & & & \\
a_{,} & a_{,} - a_{,,} & a - 2a_{,} + a_{,,} & & \\
a & a - a_{,} & a' - 2a + a_{,} & a' - 3a + 3a_{,} - a_{,,} & a'' - 4a' + 6a - 4a_{,} + a_{,,} \\
a' & a' - a & a'' - 2a' + a & a'' - 3a' + 3a - a_{,} & \\
a'' & a'' - a' & & &
\end{array}$$

ex quo sequitur:

$$b = \tfrac{1}{2}(a' - a_{,}),\ \text{quod } 2b = b_{,} - b';\ c = a' - 2a + a_{,}$$
$$d = \tfrac{1}{2}(a'' - a_{,,}) - (a' - a_{,}) = \tfrac{1}{2}(a'' - a_{,,}) - 2b,\ \text{quod } 2d = d_{,} + d';$$
$$e = a'' - 4a' + 6a - 4a_{,} + a_{,,}.$$

Est igitur $a_{,,} + 2 b_{,,} + c_{,} = a$; porro

$$b_{,,} + \tfrac{3}{2} c_{,} + \tfrac{1}{3} d_{,} - \tfrac{1}{12} e = \tfrac{2}{3}(a' - a_{,}) - \tfrac{1}{12}(a'' - a_{,,}) = \tfrac{4}{3} b - \tfrac{1}{6} d - \tfrac{1}{3} b = b - \tfrac{1}{6} d;$$

deinde $\tfrac{1}{2} c_{,} + \tfrac{1}{2} d_{,} - \tfrac{1}{24} e = \tfrac{1}{2}(a' - 2a + a_{,}) - \tfrac{1}{24} e = \tfrac{1}{2} c - \tfrac{1}{24} e$;

postea $\tfrac{1}{6} d_{,} + \tfrac{1}{12} c = \tfrac{1}{12}(a'' - a_{,,}) - \tfrac{1}{6}(a' - a_{,}) = \tfrac{1}{6} d_{,}$

et denique $\tfrac{1}{24} e = \tfrac{1}{24} e$;

qui ultimo loco inventus valor y in hanc formulam abit:

$$y = a + (b - \tfrac{1}{6} d)\, T' + (\tfrac{1}{2} c - \tfrac{1}{24} e)\, T'^2 + \tfrac{1}{6} d .\, T'^3 + \tfrac{1}{24} e .\, T'^4 + \dots.$$

quæ ita scripta, ut a, b, c, d, e in eodem ponantur loco, quo a, T', T'^2, T'^3, T'^4, ... hæc fit:

$$y = a + T'.\, b + \tfrac{1}{2} T'^2.\, c + \tfrac{1}{6}(T'^3 - T')\, d + \tfrac{1}{24}(T'^4 - T'^2)\, e + \dots \text{ seu}$$

$$y = a + T'.b + \frac{T'^2}{2}.c + \frac{T'(T'^2-1)}{2.3}d + \frac{T'^2(T'^2-1)}{2.3.4}e + \dots \text{(VII.)}$$

quæ formula apud Besselium [7.] est.

§ 9. Si porre secundum § 6. pro expressionibus

$$P = p + p'.T', \qquad Q = q + q'.T'$$

universales posueris $y = a + t.T'$, habebis $t = \frac{y-a}{T'}$, unde sequitur, formula (VII.) adhibita, t, id est p seu $q =$

$$b + \frac{T'}{2}.c + \frac{T'^2-1}{2.3}.d + \frac{T'(T'^2-1)}{2.3.4}e + \dots \text{(VIII.)}$$

quæ quidem formula apud Besselium [8.] est. Simili modo invenies expressionem, quæ Besselio est [9.]

§ 10. Valores $p' = \frac{dP}{\delta t}$, $q' = \frac{\delta Q}{\delta t}$ esse veros, per se patet. Hos igitur valores ut invenias, P et Q in (4) § 4., ratione habita temporis t, differentiandi sunt, unde:

$$p' = \frac{\cos\delta\cos(\alpha - \mathfrak{A})}{\sin\pi}.\frac{d\alpha}{dt} - \frac{\sin(\alpha-\mathfrak{A})\sin\delta}{\sin\pi}.\frac{d\delta}{dt}$$
$$- \frac{\cos\delta\sin(\alpha-\mathfrak{A})}{\sin\pi}.\cot\pi.\frac{d\pi}{dt}$$
$$q' = \frac{\cos\delta\cos D}{\sin\pi}.\frac{d\delta}{dt} + \frac{\sin D\cos(\alpha-\mathfrak{A})\sin\delta}{\sin\pi}.\frac{d\delta}{dt}$$
$$+ \frac{\cos\delta\sin D\sin(\alpha-\mathfrak{A})}{\sin\pi}.\frac{d\delta}{dt} - \left\{\frac{\sin\delta\cos D - \cos\delta\sin D\cos(\alpha-\mathfrak{A})}{\sin\pi}\right\}\cot\pi\frac{d\pi}{dt}$$

Ut porro p' et q' fiant secundæ circuli, omnia membra æquationum p' et q' definientium per radium $\omega = 206265$, qui ipse per secundas exprimitur, sunt dividenda; atque facile patet:

$$\text{(X.)}\begin{cases} p' = \dfrac{\cos\delta\cos(\alpha-\mathfrak{A})}{\omega\sin\pi}.\dfrac{d\alpha}{dt} - \dfrac{\sin\delta\sin(\alpha-\mathfrak{A})}{\omega\sin\pi}.\dfrac{d\delta}{dt} - \dfrac{p}{\omega\tan\pi}.\dfrac{d\pi}{dt} \\ q' = \dfrac{\cos\delta\sin D\sin(\alpha-\mathfrak{A})}{\omega\sin\pi}.\dfrac{d\alpha}{dt} + \left\{\dfrac{\cos\delta\cos D + \sin\delta\sin D\cos(\alpha-\mathfrak{A})}{\omega\sin\pi}\right\}.\dfrac{d\delta}{dt} \\ \qquad - \dfrac{q}{\omega\tan\pi}.\dfrac{d\pi}{dt}\end{cases}$$

quibus in formulis $\frac{d\alpha}{dt}$, $\frac{d\delta}{dt}$ et $\frac{d\pi}{dt}$ mutationes argumentorum α, δ et π resp. in una hora significant; quæ quidem formulæ Besselio in sectione octava sunt [10.]

§ 11. Cardo nonæ sectionis in eo vertitur, ut correctiones $\Delta\alpha, \Delta\delta, \Delta\pi$ et Δe^2 pro i et i' in formulam (VI.) restituantur. Erat enim secundum (5*) § 6.

$$p'.i - q'.i' = a\Delta\alpha + b\Delta\delta + c\Delta\pi + d\Delta e^2$$
$$q'.i + p'.i' = a'\Delta\alpha + b'\Delta\delta + e'\Delta\pi + d'\Delta e^2,$$

ubi, secundum ea quæ ad § 5, ex. adnotavimus, coëfficientes correctionum $\Delta\alpha, \Delta\delta, \Delta\pi$ et Δe^2 sunt partialia differentialium quota quantitatum $P - u$ et $Q - v$, si respexeris α, δ, π et e^2; igitur

$$a = \frac{dP}{d\alpha}, b = \frac{dP}{d\delta}, c = \frac{dP}{d\pi}, d = -\frac{du}{de^2},$$
$$a' = \frac{dQ}{d\alpha}, b' = \frac{dQ}{d\delta}, c' = \frac{dQ}{d\pi}, d' = -\frac{dv}{de^2}.$$

Hæ expressiones ex differentiationibus æquationum (4.) § 4. nascuntur sic:

$$a = \frac{\cos\delta\cos(\alpha - A)}{\sin\pi}, \qquad a' = \frac{\cos\delta\sin D\sin(\alpha - A)}{\sin\pi},$$
$$b = -\frac{\sin\delta\sin(\alpha - A)}{\sin\pi}, \qquad b' = \frac{\cos\delta\cos D + \sin\delta\sin D\cos(\alpha - A)}{\sin\pi},$$
$$c = -\frac{P}{\tan\pi}, \qquad c' = -\frac{Q}{\tan\pi},$$
$$d = -\frac{du}{de^2}, \qquad d' = -\frac{dv}{de^2}.$$

Si parvas quantitates $\alpha - A$, $\delta - D$ neglexeris, ex quo vitium alicujus momenti plane non poterit oriri, propterea quod $\Delta\alpha$, $\Delta\delta$, $\Delta\pi$ et Δe^2 ipsæ sunt minimæ quantitates; hæ simplicissimæ expressiones fient:

$$a = \frac{\cos\delta}{\sin\pi}, b = o, c = -\frac{P}{\tan\pi}, d = -\frac{du}{de^2};$$
$$a' = o, b' = \frac{1}{\sin\pi}, c' = -\frac{Q}{\tan\pi}, d' = -\frac{dv}{de^2}.$$

Quodsi hos valores pro a, b, c; a', b', c', substitueris in æquationibus i et i' definientibus, quæ occurrunt §. nostra ineunt, et si pro p' et q' sumseris valoris ex (5. b.) § 6.: sequetur

$$n\sin N.\, i - n\cos N.\, i' = \frac{\cos\delta}{\sin\pi}.\,\Delta\alpha - P\cot\pi.\,\Delta\pi - \frac{du}{de^2}.\,\Delta e^2$$
$$n\cos N.\, i + n\sin N.\, i' = \frac{1}{\sin\pi}.\,\Delta\delta - Q\cot\pi.\,\Delta\pi - \frac{dv}{de^2}.\,\Delta e^2.$$

Quarum æquationem si prior per $\sin N$ atque posterior per $\cos N$ multiplicata, et deinde illa ad hanc addita est: fiet

$$n.\, i = \frac{\cos\delta\sin N}{\sin\pi}.\,\Delta\alpha + \frac{\cos N}{\sin\pi}.\,\Delta\delta - (P\sin N + Q\cos N)\cot\pi.\,\Delta\pi$$
$$- \left(\frac{du}{de^2}.\sin N + \frac{dv}{de^2}.\cos N\right)\Delta e^2.$$

Sin autem priorem per $\cos N$ multiplicatam a posteriore per $\sin N$ multiplicata subtraxeris, erit:

$$n.\, i' = -\frac{\cos\delta\cos N}{\sin\pi}.\,\Delta\alpha + \frac{\sin N}{\sin\pi}.\,\Delta\delta - (Q\sin N - P\cos N)\cot\pi.\,\Delta\pi$$
$$- \left(\frac{du}{de^2}.\sin N - \frac{du}{de^2}.\cos N\right)\Delta e^2.$$

In his æquationibus quantitates $\Delta\alpha$, $\Delta\delta$, $\Delta\pi$ per radii partes sunt expressæ, quæ, ut secundæ fiant, dividendæ sunt per ω, § 10, commemoratum. Unde fit:

$$i = \frac{s\cos\delta\sin N}{n\omega\sin\pi}.\,\Delta\alpha + \frac{s\cos N}{n\omega\sin\pi}.\,\Delta\delta - \frac{s(P\sin N + Q\cos N)}{n\omega\sin\pi}.\cos\pi.\,\Delta\pi$$
$$- \frac{s\left(\frac{du}{de^2}.\sin N + \frac{dv}{de^2}.\cos N\right)}{n\omega\sin\pi}.\,\omega\sin\pi.\,\Delta e^2$$

$$i' = -\frac{s\cos\delta\cos N}{n\,\omega\sin\pi}.\Delta\alpha + \frac{s\sin N}{n\,\omega\sin\pi}.\Delta\delta - \frac{s\,(Q\sin N - P\cos N)}{n\,\omega\sin\pi}.\cos\pi.\Delta\pi$$
$$-\frac{s\left(\frac{d\,v}{d\,e^2}\sin N - \frac{d\,u}{d\,e^2}.\cos N\right)}{n\,\omega\sin\pi}.\,\omega\sin\pi\,.\,\Delta\,e^2.$$

Quid denotet hoc loco signum s, in practica parte infra docebimus. Si brevitatis causa $\frac{s}{\omega\,n\sin\pi} = h$ ponitur, formulæ existunt:

$$i = h\sin N\cos\delta\,.\,\Delta\,\alpha + h\cos N\,.\,\Delta\,\delta - h\cos\pi\,.\,\Delta\,\pi\,(P\sin N + Q\cos N)$$
$$-h\,\omega\sin\pi\,.\,\Delta\,e^2\left(\frac{d\,u}{d\,e^2}\sin N + \frac{d\,v}{d\,e^2}\cos N\right),$$
$$i' = -h\cos N\cos\delta\,.\,\Delta\,\alpha + h\sin N\,.\,\Delta\,\delta + h\cos\pi\,.\,\Delta\,\pi\,(P\cos N - Q\sin N)$$
$$+h\,\omega\sin\pi\,.\,\Delta\,e^2\left(\frac{d\,u}{d\,e^2}\cos N - \frac{d\,v}{d\,e^2}\sin N\right);$$

quas etiam Besselius habet. Jam vero si i et $\frac{\sin\psi}{h}$. atque i' et $\frac{\cos\psi}{h}$ inter se multiplicantur, deinde adduntur, hæc expressio videbitur:

$$\left(i + \frac{i'}{\tan\psi}\right)\frac{\sin\psi}{h} = -\cos(N+\psi)\cos\delta\,.\,\Delta\,\alpha + \sin(N+\psi)\,.\,\Delta\,\delta$$
$$+\cos\pi\,\mathfrak{A}\,.\,\Delta\,\pi + \omega\sin\pi\,\mathfrak{B}\,.\,\Delta\,e^2\;\ldots\;(6.)$$

in qua brevitatis causa posuimus

$$\left.\begin{aligned} &P\cos(N+\psi) - Q\sin(N+\psi) = \mathfrak{A} \\ &\frac{d\,u}{d\,e^2}\cos(N+\psi) - \frac{d\,v}{d\,e^2}\sin(N+\psi) = \mathfrak{B} \end{aligned}\right\}\ldots\;(6.\,a.).$$

§ 12. Expressiones $\mathfrak{A}$ et $\mathfrak{B}$ definientes etiam alia ratione possunt transformari. Nam secundum (5.) et (5. b.) erat $P = p + n\sin N\,.\,T'$, $Q = q + n\cos N\,.\,T'$, itaque

$$\mathfrak{A} = (p + n\sin N.\,T')\cos(N+\psi) - (q + n\cos N.\,T')\sin(N+\psi)$$
$$= -(q\sin N - p\cos N)\cos\psi - (p\sin N + q\cos N + n\,T')\sin\psi.$$

Jam ponas

$$q\sin N - p\cos N = x\,,\; n\,\tau = n\,T - s\,p\sin N - s\,q\cos N;$$

porro positumerat

$$s\,T' = t - d - T,\text{ unde } T = t - d - s\,T',$$

ergo

$$n\,\tau = n\,t - n\,d - s\,n\,T' - s\,p\sin N - s\,q\cos N,$$

et ex hac formula derivabitur

$$n\,T' = \frac{n}{s}\,(t - d - \tau) - p\sin N - q\cos N;$$

igitur

$$p\sin N + q\cos N + n\,T' = \frac{n}{s}(t - d - \tau,)$$

et demum

$$\mathfrak{A} = -\left\{x\cos\psi + \frac{n}{s}\,(t - d - \tau)\sin\psi\right\}\ldots(7.)$$

Transformatio expressionis $\mathfrak{B}$ pendet ab inventione differentialium quotarum $\frac{d\,u}{d\,e^2}$, $\frac{d\,v}{d\,e^2}$, quæ ipsæ iterum a quantitatibus $r \sin \phi'$, $r \cos \phi'$ pendent. Quum $\frac{d\,u}{d\,e^2}$ et $\frac{d\,v}{d\,e^2}$, duplici modo definiri possint, etiam expressionem $\mathfrak{B}$ duplici modo poteris transformare: valor ϕ enim, qui inest in u et v, aut declinationem aut latitudinem quincti verticis denotat. Utrumque igitur accuratius inquiramus. Primum pro valoribus $r \sin \phi'$ et $r \cos \phi'$ expressiones sunt inveniendæ, in quibus e^2 et ϕ continentur. Æquationes

$$y^2 = B^2 - \frac{B^2}{\mathfrak{A}^2} x^2,\ x = r \cos \phi',\ y = r \sin \phi',\ \mathrm{tg}\, \phi' = \frac{B^2}{\mathfrak{A}^2} \tan \phi$$

adesse constat. Valores x et y si in primam substitueris æquationem, hæc existit æquatio:

$$r^2 \sin \phi'^2 = B^2 - \frac{B^2}{\mathfrak{A}^2} r^2 \cos \phi'^2,$$

ex qua, si diviseris per $\cos \phi'^2$ et $\sin \phi'^2$, sequitur

$$r^2 \tan \phi'^2 = \frac{B^2}{\cos \phi'^2} - \frac{B^2}{\mathfrak{A}^2} r^2, \qquad r^2 = \frac{B^2}{\sin \phi'^2} - \frac{B^2}{\mathfrak{A}^2} \cdot \frac{r^2}{\tan \phi'^2}.$$

Si deinde in his æquationibus pro $\tan \phi'$ posueris valorem supra inventum, et recordatus fueris $\frac{\mathfrak{A}^2 - B^2}{\mathfrak{A}^2} = e^2$ positum a nobis esse, post hæc omnia has invenies expressiones:

$$r \cos \phi' = \frac{\cos \phi}{\sqrt{1 - e^2 \sin \phi^2}}, \qquad r \sin \phi' = \frac{(1 - e^2) \sin \phi}{\sqrt{1 - e^2 \sin \phi^2}} \ldots (7.\ \mathrm{a.}).$$

Ex utraque expressione per differentiationem habebis:

$$\frac{d\,.\, r \cos \phi}{d\,.\, e^2} = r \cos \phi' . \frac{\sin \phi^2}{2\,(1 - e^2 \sin \phi^2)}$$

$$\frac{d\,.\, r \sin \phi'}{d\,.\, e^2} = r \sin \phi' . \frac{\sin \phi^2}{2\,(1 - e^2 \sin \phi^2)} - \frac{\sin \phi}{\sqrt{1 - e^2 \sin \phi^2}};$$

ex posteriore vero æquatione in (7. a.) sequetur:

$$\frac{\sin \phi}{\sqrt{1 - e^2 \sin \phi^2}} = \frac{r \sin \phi'}{1 - e^2}$$

$$\text{itaque } \frac{d\,.\, r \cos \phi'}{d\,.\, e^2} = r \cos \phi' . \frac{r^2 \sin \phi'^2}{2\,(1 - e^2)^2},$$

$$\frac{d\,.\, r \sin \phi'}{d\,.\, e^2} = r \sin \phi' . \frac{r^2 \sin \phi'^2}{2\,(1 - e^2)^2} - \frac{r \sin \phi'}{1 - e^2}.$$

Si brevitatis causa $\frac{r \sin \phi'}{1 - e^2} = \beta$ ponitur, erit:

$$\frac{d\,.\, r \cos \phi}{d\,.\, e^2} = \tfrac{1}{2} \beta^2 r \cos \phi',$$

$\frac{d\,.\, r \sin \phi'}{d\,.\, e^2} = \frac{1}{2} \beta^2\, r \sin \phi' - \beta$; ut in Besselii dissertatione.

Secundum autem § 4 (4.) est

$$u = r \cos \phi' \sin (\mu - \mathfrak{A}),\ v = r \sin \phi' \cos D - r \cos \phi' \sin D \cos (\mu - \mathfrak{A}),$$

ergo

$$\frac{d\,u}{d\,e^2} = \left(\frac{d\,.\, r \cos \phi'}{d\,.\, e^2} \right) \sin (\mu - \mathfrak{A}) = \tfrac{1}{2} \beta^2\, r \cos \phi' \sin (\mu - \mathfrak{A}),$$

id est

$$\frac{d\,u}{d\,e^2} = \tfrac{1}{2}\,\beta^2.\,u;$$

atque

$$\frac{d\,v}{d\,e^2} = \left(\frac{d\,.\,r\sin\phi'}{d\,.\,e^2}\right)\cos D - \left(\frac{d\,.\,r\cos\phi'}{d\,.\,e^2}\right)\sin D\cos(\mu - \mathfrak{A})$$
$$= \tfrac{1}{2}\,\beta^2\,r\left\{\sin\phi'\cos D - \cos\phi'\sin D\cos(\mu - \mathfrak{A})\right\} - \beta\cos D,$$

id est

$$\frac{d\,v}{d\,e^2} = \tfrac{1}{2}\,\beta^2.\,v - \beta\cos D$$

Quos valores modo interventos differentialium $\frac{d\,u}{d\,e^2}$ et $\frac{d\,v}{d\,e^2}$, si in expressionem quantitatis $\mathfrak{B}$ [§ 11. (6 a.)] posueris, hanc formulam habebis:

$$\mathfrak{B} = \tfrac{1}{2}\,\beta^2\,u\cos(N+\psi) - (\tfrac{1}{2}\,\beta^2\,v - \beta\cos D)\sin(N+\psi)$$
$$= \tfrac{1}{2}\,\beta^2\left\{u\cos(N+\psi) - v\sin(N+\psi)\right\} + \beta\cos D\sin(N+\psi)\,\ldots\,(7.\text{ b.})$$

Facile igitur intelligitur

$$u\cos(N+\psi) - v\sin(N+\psi) = \left\{P\cos(N+\psi) - Q\sin(N+\psi)\right\}$$
$$- \left\{(P-u)\cos(N+\psi) - (Q-v)\sin(N+\psi)\right\},$$

hoc est

$$= \mathfrak{A} - \left\{(P-u)\cos(N+\psi) - (Q-v)\sin(N+\psi)\right\}.$$

Sed ex (5.) § 6. derivabitur

$$P - u = p - u + p'.\,T', \qquad Q - v = q - v + q'.\,T';$$

et, si usurpaveris expressiones in (5. b.), habebis

$$P - u = m\sin M + n\sin N.\,T'$$
$$Q - v = m\cos M + n\cos N.\,T';$$

ergo

$$(P-u)\cos(N+\psi) - (Q-v)\sin(N+\psi) =$$
$$(m\sin M + n\sin N.\,T')\cos(N+\psi) - (m\cos M + n\cos N.\,T')\sin(N+\psi)$$
$$= m\left\{\sin M\cos(N+\psi) - \cos M\sin(N+\psi)\right\}$$
$$+ n\,T'\left\{\sin N\cos(N+\psi) - \cos N\sin(N+\psi)\right\}$$
$$= m\sin(M - N - \psi) - n\,T'.\sin\psi;$$

atque si pro T' valorem approximatum ex (V.) § 6. scilicet

$$T' = -\frac{m\cos(M - N - \psi)}{n\cos\psi}$$

posueris, $= m\sin(M - N - \psi) + \frac{m\cos(M-N-\psi)}{u\cos\psi}\,n\sin\psi = \frac{m\sin(M-N)}{\cos\psi}$, hoc est secundum (5. c.) § 6., $= k$; ergo $u\cos(N+\psi) - v\sin(N+\psi) = \mathfrak{A} - k$, atque sic, si hoc substitueris in (7. b.), $\mathfrak{B} = \tfrac{1}{2}\,\beta^2\,(\mathfrak{A} - k) + \beta\cos D\sin(N+\psi)$.

Si denique (id quod solum restat) pro $\mathfrak{A}$ ejus valor ex (7.) ponitur, erit:

$$\mathfrak{B} = \tfrac{1}{2}\,\beta^2\left\{-\varkappa\cos\psi - \frac{n}{s}(t - d - \tau)\sin\psi - k\right\} + \beta\cos D\sin(N+\psi)$$
$$= -\tfrac{1}{2}\,\beta^2\left\{\varkappa\cos\psi + \frac{n}{s}(t - d - \tau)\sin\psi + k\right\} + \beta\cos D\sin(N+\psi)\ldots(7.\text{ c.})$$

§ 13. Ex (6.) § 11. facillime deducitur:

$$i+\frac{i'}{\tan\psi}=-\frac{h\cos(N+\psi)\cos\delta}{\sin\psi}.\Delta\alpha+\frac{h\sin(N+\psi)}{\sin\psi}.\Delta\delta+\frac{h\cos\pi}{\sin\psi}.\mathfrak{A}.\Delta\pi+\frac{h\,\omega\sin\pi}{\sin\psi}.\mathfrak{B}.\Delta e^2$$

Hic summæ $i+\frac{i'}{\tan\psi}$ valor si ponitur in expressionem d, [§ 6. (VI.)] hæc existit æquatio:

$$d=t-T+\frac{m\,s\cos(M-N-\psi)}{n\cos\psi}-\frac{h\cos(N+\psi)\cos\delta}{\sin\psi}.\Delta\alpha+\frac{h\sin(N+\psi)}{\sin\psi}.\Delta\delta$$
$$+\frac{h\cos\pi}{\sin\psi}.\mathfrak{A}.\Delta\pi+\frac{h\,\omega\sin\pi}{\sin\psi}.\mathfrak{B}.\Delta e^2+\frac{s.\Delta k}{n\sin\psi}\ldots.(7.\ d.)$$

et, si pro $\mathfrak{A}$, $\mathfrak{B}$ eorum valores (7.), (7. c.) derivati substituuntur, erit

$$d=t-T+\frac{m\,s\cos(M-N-\psi)}{n\cos\psi}-\frac{h\cos(N+\psi)\cos\delta}{\sin\psi}.\Delta\alpha+\frac{h\sin(N+\psi)}{\sin\psi}.\Delta\delta$$
$$-\frac{h\cos\pi}{\sin\psi}.\Delta\pi\left\{x\cos\psi+\frac{n}{s}(t-d-\tau)\sin\psi\right\}+\frac{\Delta k}{n\sin\psi}$$
$$-\frac{h\,\omega\sin\pi}{\sin\psi}.\Delta e^2\left\{\tfrac{1}{2}\beta^2\left[x\cos\psi+\frac{n}{s}(t-d-\tau)\sin\psi+k\right]-\beta\cos D\sin(N+\psi)\right\};$$

vel, ut usui aptior sit,

$$d=t-T+\frac{m\,s\cos(M-N-\psi)}{n\cos\psi}-\frac{h\cos(N+\psi)}{\sin\psi}\cos\delta.\Delta\alpha$$
$$+\frac{h\sin(N+\psi)}{\sin\psi}.\Delta\delta+h.\frac{1}{\sin\psi}.\omega\sin\pi.\Delta k$$
$$-h\cos\pi.\Delta\pi\left\{\frac{x}{\tan\psi}+\frac{n}{s}(t-d-\tau)\right\}$$
$$-h\,\omega\sin\pi.\Delta e^2\left\{\tfrac{1}{2}\beta^2\left[\frac{x}{\tan\psi}+\frac{n}{s}(t-d-\tau)+\frac{k}{\sin\psi}\right]-\frac{\beta V}{\sin\psi}\right\}\ldots.(\mathrm{XI.})$$

Hæc expressio Besselio in sectione (10.) est [11.]. Restat, ut moneamus, signum V, brevitatis causa positum, denotare $\cos D\sin(N+\psi)$, ad quod signum infra redeundum erit.

§ 14. Si pro rectascensionibus declinationibusque in calculo longitudines et latitudines adhibentur, cardo rei in eo versatur, ut latitudo φ' et longitudo μ puncti verticis ex rectascensione (μ) et declinatione (φ') data definiatur. Fingas tibi (Fig. 3.) P et E polos æquatoris et eclipticæ, Z punctum verticis, V intersectionem eclipticæ in æquatorem; erit $\angle EPZ=90°+(\mu)$, arcus $PE=\varepsilon$ obliquitas eclipticæ; $Vp=(\mu)$, $Ve=\mu$, $Zp=(\varphi')$, $Ze=\varphi'$. Jam in $\Delta\,VZe$ est $\cos VZ=\cos\mu\cos\varphi'$, et in $\Delta\,VZp$ est $\cos VZ=\cos(\mu)\cos(\varphi')$; ergo $\cos\mu\cos\varphi'=\cos(\mu)\cos(\varphi')\ldots.(8.)$, porro in $\Delta\,PEZ$ est $\sin\varphi'=\sin(\varphi')\cos\varepsilon-\cos(\varphi')\sin(\mu)\sin\varepsilon\ldots(8.\ a.)$ Æquatio (8.) etiam ita scribi potest:

$$\cos\varphi'\sqrt{1-\sin\mu^2}=\cos(\mu)\cos(\varphi'),$$

igitur

$$\cos\varphi'^2-\cos\varphi'^2\sin\mu^2=\cos(\mu)^2\cos(\varphi')^2,$$

unde

$$\cos\varphi'\sin\mu=\sqrt{1-\sin\varphi'^2-\cos(\varphi')^2\cos(\mu)^2},$$

Hac in formula igitur si pro $\sin\varphi'^2$ valor ex (8. a.) ponitur, reductionibus quibusdam haud difficilibus adhibitis, hæc existit æquatio:

$$\cos\varphi' \sin\mu = \sin(\phi') \sin\varepsilon + \cos(\varphi') \sin(\mu) \cos\varepsilon \dots (8.\ b.)$$

Nota.—In nona Besselianæ dissertationis sectione, p. 136, legendum est $\cos\phi' \sin\mu$ pro $\cos\phi' \sin\mu'$.

Erat autem secundum (4.) § 4.

$$u = r\cos A \cos\varphi' \sin\mu - r\sin A \cos\varphi' \cos\mu$$
$$v = r\cos D \sin\varphi' - r\sin D \cos A \cos\varphi' \cos\mu - r\sin D \sin A \cos\phi' \sin\mu\,;$$

si igitur pro $\cos\phi' \sin\mu$, $\cos\phi' \cos\mu$ et $\sin\phi'$ valores modo deducti substituuntur erit:

$$u = r\sin(\varphi') \sin\varepsilon \cos A + r\cos(\varphi') \left\{\cos A \sin(\mu) \cos\varepsilon - \sin A \cos(\mu)\right\}$$
$$v = r\sin(\varphi') \left\{\cos D \cos\varepsilon - \sin D \sin\varepsilon \sin A\right\}$$
$$- r\cos(\varphi') \left\{\sin(\mu) \left[\cos D \sin\varepsilon + \sin D \cos\varepsilon \sin A\right] + \cos(\mu) \sin D \cos A\right\}.$$

Nota.—In Besselii dissertatione desideratur in ultimo membro æquationis v definientis signum $+$.

Secundum æquationis, quæ § 12. leguntur, habebis analogas:

$$\beta = \frac{r\sin(\varphi')}{1-e^2}, \frac{d\,.\,r\cos(\varphi')}{d\,.\,e^2} = \tfrac{1}{2}\beta^2 r\cos(\varphi'),$$
$$\frac{d\,.\,r\sin(\varphi')}{d\,.\,e^2} = \tfrac{1}{2}\beta^2 r\sin(\varphi') - \beta\,;$$

ergo

$$\frac{du}{de^2} = \sin\varepsilon \cos A \left\{\tfrac{1}{2}\beta^2 r\sin(\varphi') - \beta\right\} + \left\{\cos A \sin(\mu) \cos\varepsilon - \sin A \cos(\mu)\right\} + \tfrac{1}{2}\beta^2 r\cos(\varphi')$$
$$= \tfrac{1}{2}\beta^2 \left\{r\sin(\varphi') \sin\varepsilon \cos A + r\cos(\varphi') \left[\cos A \sin(\mu) \cos\varepsilon - \sin A \cos(\mu)\right]\right\} - \beta \sin\varepsilon \cos A,$$

hoc est

$$\frac{du}{de^2} = \tfrac{1}{2}\beta^2 u - \beta \sin\varepsilon \cos A \dots (9.)$$

atque

$$\frac{dv}{de^2} = (\cos D \cos\varepsilon - \sin D \sin\varepsilon \sin A) \left[\tfrac{1}{2}\beta^2 r\sin(\varphi') - \beta\right]$$
$$-\left\{\sin(\mu) \left[\cos D \sin\varepsilon + \sin D \cos\varepsilon \sin A\right] + \cos(\mu) \sin D \cos A\right\} \tfrac{1}{2}\beta^2 .\, r\cos(\varphi')$$
$$= \tfrac{1}{2}\beta^2 \left\{r\sin(\varphi') \left[\cos D \cos\varepsilon - \sin D \sin\varepsilon \sin A\right]\right.$$
$$\left. - r\cos(\varphi') \left[(\cos D \sin\varepsilon + \sin D \cos\varepsilon \sin A) \sin(\mu) + \sin D \cos A \cos(\mu)\right]\right\}$$
$$- \beta (\cos D \cos\varepsilon - \sin D \sin\varepsilon \sin A),$$

hoc est:

$$\frac{dv}{de^2} = \tfrac{1}{2}\beta^2 v - \beta (\cos D \cos\varepsilon - \sin D \sin\varepsilon \sin A) \dots (9.\ a.)$$

Qui valores si posueris in æquationem (6. a.) § 11., erit

$$\mathfrak{B} = (\tfrac{1}{2}\beta^2 u - \beta \sin\varepsilon \cos A) - \left\{\tfrac{1}{2}\beta^2 v - \beta (\cos D \cos\varepsilon - \sin D \sin\varepsilon \sin A)\right\} \sin(N+\psi)$$
$$= \tfrac{1}{2}\beta^2 \left\{u\cos(N+\psi) - v\sin(N+\psi)\right\} - \beta \sin\varepsilon \cos A \cos(N+\psi)$$
$$+ \beta (\cos D \cos\varepsilon - \sin D \sin\varepsilon \sin A) \sin(N+\psi)\,;$$

id est:

$$\mathfrak{B} = \tfrac{1}{2}\beta^2 (\mathfrak{A} - k) - \beta \sin\varepsilon \cos A \cos(N+\psi) + \beta (\cos D \cos\varepsilon - \sin D \sin\varepsilon \sin A) \sin(N+\psi),$$

igitur etiam secundum (7.) § 12.

$$\mathfrak{A} = -\Big\{[x \cos\psi + \frac{n}{s}(t-d-\tau)\sin\psi + k]\tfrac{1}{2}\beta^2 + \beta\sin\varepsilon\cos\mathcal{A}\cos(N+\psi)$$
$$-\beta(\cos D\cos\varepsilon - \sin D\sin\varepsilon\sin\mathcal{A})\sin(N+\psi)\Big\}\ldots\ldots(9.\ \mathrm{b.})$$

Itaque si in (7. d.) § 13. pro $\mathfrak{A}$ et $\mathfrak{B}$ eorum valores ex (7.) § 12. et ex (9. b.) § 14. substitueris, redibit formula (XI.), si nimirum ibi ponitur:

$$V = (\cos D\cos\varepsilon - \sin D\sin\varepsilon\sin\mathcal{A})\sin(N+\psi) - \sin\varepsilon\cos\mathcal{A}\cos(N+\psi)\ldots\ldots(9.\ \mathrm{c.})$$

§ 15. Formula (XI.) etiam hac ratione scribi potest:

$$d = t - T + \frac{m\,s\cos(M-N-\psi)}{n\cos\psi} + h(\sin N\cos\delta.\Delta\alpha + \cos N.\Delta\delta)$$
$$+\frac{h}{\tan\psi}(-\cos N\cos\delta.\Delta\alpha + \sin N.\Delta\delta - k\cos\pi.\Delta\pi)$$
$$+\frac{h}{\sin\psi}(\omega\sin\pi.\Delta k) - \frac{h\,n}{s}(t-d-\tau)(\cos\pi.\Delta\pi)$$
$$-h\Big\{\tfrac{1}{2}\beta^2\Big[\frac{x}{\tan\psi} + \frac{n}{s}(t-d-\tau) + \frac{k}{\sin\psi}\Big] - \frac{\beta V}{\sin\psi}\Big\}(\omega\sin\pi.\Delta e^2)\ldots(10.)$$

Si igitur ponitur

$$\left.\begin{array}{l} +\sin N\cos\delta.\Delta\alpha + \cos N.\Delta\delta = \varepsilon \\ -\cos N\cos\delta.\Delta\alpha + \sin N.\Delta\delta - k\cos\pi.\Delta\pi = \zeta \\ \omega\sin\pi.\Delta k = \eta,\ \cos\pi.\Delta\pi = \theta,\ \omega\sin\pi.\Delta e^2 = i \end{array}\right\}\ldots(11.)$$

expressiones (10) forma hæc erit:

$$d = t - T + \frac{m\,s\cos(M-N-\psi)}{n\cos\psi} + h\varepsilon + \frac{h}{\tan\psi}\zeta + \frac{h}{\sin\psi}\eta - hE.\theta - hF.i\ldots(\mathrm{XII.}),$$

ubi

$$E = \frac{n}{s}(t-d-\tau)$$
$$F = \left(\frac{x}{\tan\psi} + E + \frac{x}{\sin\psi}\right)\tfrac{1}{2}\beta^2 - \frac{V\beta}{\sin\psi}\ldots(\mathrm{n.})$$

Expressio (XII.), apud Besselium [12.], nunc in calculo numerico formam commodiorem et ad perlustrandum faciliorem habet, quam quæ ei identica est (XI.).

§ 16. In duodecima sectione ad æquationem fundamentalem (II.) reversus, ejus formam usui aptissimam Besselius definire studet, id ponens utrique corpori cœlesti esse diametrum et parallaxin. Hanc nimirum dixit esse formam ad inveniendum difficiliorem, quam ubi de fixæ occultatione esset sermo. Quæ nunc dicturi sumus, ejus illustrabunt solvendi rationem.

§ 17. In æquatione (3.) secundum quadratum ita comparatum putes, ut possit in nihilum redigi, ita ut sit

$$e\sin u - f\cos u\cos v + g\cos u\sin v = 0\ldots(13.)$$

Quod autem fit, si cogites, expressionem

$$(a\,b' - a'\,b)(c'\sin\pi - c\sin\pi') - (a\,c' - a'\,c)(b'\sin\pi - b\sin\pi') + (b\,c' - b'\,c)(a'\sin\pi - a\sin\pi') = 0$$

esse, itaque etiam

$$e(c'\sin\pi - c\sin\pi') - f(b'\sin\pi - b\sin\pi') + g(a'\sin\pi - a\sin\pi') = 0\ldots(13.\ \mathrm{a.})$$

Æquatio (13.) non mutatur, si d et a pro u et v substitueris et deinde unum quodque membrum æquationis per novam quantitatem ignotam G multiplicaveris; quibus enim factis habebis

$$e\,G \sin d - f\,G \cos d \cos a + g\,G \cos d \sin a = 0 \ldots\ldots (13.\ b.)$$

Igitur si comparaveris (13. b.) cum (13. a.) habebis tres hasce æquationes, quæ definiunt G, d et a:

$$\left.\begin{aligned} G \sin d &= c' \sin \pi - c \sin \pi' \\ G \cos d \cos a &= b' \sin \pi - b \sin \pi' \\ G \cos d \sin a &= a' \sin \pi - a \sin \pi' \end{aligned}\right\} \ldots (13.\ c.)$$

Quare mutata æquatio (3.)

$$e^2 + f^2 + g^2 = (e \cos u + f \sin u \cos v - g \sin u \sin v)^2 + (f \sin v + g \cos v)^2$$

hanc accipiet formam:

$$\begin{aligned} &(a\,b' - a'\,b)^2 + (a\,c' - a'\,c)^2 + (b\,c' - b'\,c)^2 = \\ &[(a\,b' - a'\,b) \cos d + (a\,c' - a'\,c) \sin d \cos a - (b\,c' - b'\,c) \sin d \sin a]^2 \\ &\quad + [(a\,c' - a'\,c) \sin a + (b\,c' - b'\,c) \cos a]^2 \ldots\ldots (13.\ d.) \end{aligned}$$

§ 18. Si autem in (13. c.) pro a, b, c, a', b', c' earum valores ex sex prioribus æquationibus § 3. substitueris, accipies:

$$\left.\begin{aligned} G \sin d &= \sin \pi \sin D - \sin \pi' \sin \delta \\ G \cos d \cos a &= \sin \pi \cos D \cos A - \sin \pi' \cos \delta \cos \alpha \\ G \cos d \sin a &= \sin \pi \cos D \sin A - \sin \pi' \cos \delta \sin \alpha \end{aligned}\right\} \ldots\ldots (XIV.)$$

quæ efficiunt systema illud Besselii [14.]. Porro si in (13. c.) pro a, b, c, a', b', c' earum valores ex sex posterioribus æquationibus § 3, substitueris, accipies:

$$\left.\begin{aligned} G \sin d &= \Delta' \sin \pi \sin D' - \Delta \sin \pi' \sin \delta' \\ G \cos d \cos a &= \Delta' \sin \pi \cos D' \cos A' - \Delta \sin \pi' \cos \delta' \cos \alpha' \\ G \cos d \sin a &= \Delta' \sin \pi \cos D' \sin A' - \Delta \sin \pi' \cos \delta' \sin \alpha' \end{aligned}\right\} \ldots\ldots (14.).$$

Ultima in Besseliana dissertatione æquatio falsa est:

$$G \cos d \sin a = \Delta' \sin \pi \cos D' \cos A' - \Delta \sin \pi' \cos \delta' \sin \alpha'.$$

Si in (XIV.) prima per secundam, et secunda per tertiam dividuntur, existunt expressiones:

$$\frac{\tan d}{\cos a} = \frac{\sin \pi \sin D - \sin \pi' \sin \delta}{\sin \pi \cos D \cos A - \sin \pi' \cos \delta \cos \alpha} \ldots\ldots (14.\ a.)$$

$$\frac{\cos a}{\sin a} = \frac{\sin \pi \cos D \cos A - \sin \pi' \cos \delta \cos \alpha}{\sin \pi \cos D \sin A - \sin \pi' \cos \delta \sin \alpha} \ldots\ldots (14.\ b.)$$

ex quibus, eliminatis nominatoribus, sequuntur

$$(\cos D \cos A \tan d - \sin D \cos a) \sin \pi = (\cos \delta \cos \alpha \tan d - \sin \delta \cos a) \sin \pi'$$
$$(\cos D \sin A \cos a - \cos D \cos A \sin a) \sin \pi = (\cos \delta \sin \alpha \cos a - \cos \delta \cos \alpha \sin a) \sin \pi';$$

et si æquationem primam per secundam diviseris, hæc æquatio existet:

$$\frac{\cos A \tan d - \tan D \cos a}{\sin (A - a)} = \frac{\cos \alpha \tan d - \tan \delta \cos a}{\sin (\alpha - a)}$$

ex qua, nominatoribus eliminatis, sequitur Besseliana æquatio conditionis [15]:

$$\tan \delta \sin (A - a) - \tan D \sin (\alpha - a) + \tan d \sin (\alpha - A) = 0 \ldots\ldots (XV.)$$

§ 19. Porro, valoribus a, b, c, a', b', c' positis, invenitur

$$\begin{aligned} a\,b' - a'\,b &= \cos\delta\cos D\sin(\alpha - \mathcal{A}) \\ &- r\cos\varphi'\sin\mu\,(\sin\pi\cos D\cos\mathcal{A} - \sin\pi'\cos\delta\cos\alpha) \\ &+ r\cos\varphi'\cos\mu\,(\sin\pi\cos D\sin\mathcal{A} - \sin\pi'\cos\delta\sin\alpha) \\ &= \cos\delta\cos D\sin(\alpha - \mathcal{A}) - r\cos\varphi'\,G\,.\cos d\sin(\mu - a); \end{aligned}$$

deinde

$$\begin{aligned} a\,c' - a'\,c &= \cos\delta\sin D\sin\alpha - r\cos\varphi'\sin\mu\,(\sin\pi\sin D - \sin\pi'\sin\delta) \\ &- \cos D\sin\delta\sin\mathcal{A} + r\sin\varphi'\,(\cos D\sin\mathcal{A}\sin\pi - \cos\delta\sin\alpha\sin\pi') \\ &= \cos\delta\sin D\sin\alpha - \cos D\sin\delta\sin\mathcal{A} - G\,(r\cos\varphi'\sin d\sin\mu - r\sin\varphi'\cos d\sin a); \end{aligned}$$

et denique

$$\begin{aligned} b\,c' - b'\,c &= \cos\delta\sin D\cos\alpha - r\cos\varphi'\cos\mu\,(\sin\pi\sin D - \sin\pi'\sin\delta) \\ &- \cos D\sin\delta\cos\mathcal{A} + r\sin\varphi'\,(\cos D\cos\mathcal{A}\sin\pi - \cos\delta\cos\alpha\sin\pi') \\ &= \cos\delta\sin D\cos\alpha - \cos D\sin\delta\cos\mathcal{A} - G\,(r\cos\varphi'\sin d\cos\mu - r\sin\varphi'\cos d\cos a). \end{aligned}$$

Est igitur:

$$\begin{aligned} &(a\,b' - a'\,b)\cos d + (a\,c' - a'\,c)\sin d\cos a - (b\,c' - b'\,c)\sin d\sin a = \\ &- \sin\delta\cos D\sin d\sin(\mathcal{A} - a) + \cos\delta\sin D\sin d\sin(\alpha - a) \\ &+ \cos\delta\cos D\cos d\sin(\alpha - \mathcal{A}) - G\,.\,r\cos\varphi'\sin(\mu - a)\ldots\ldots(15.) \end{aligned}$$

atque

$$\begin{aligned} &(a\,c' - a'\,c)\sin a + (b\,c' - b'\,c)\cos a = \cos\delta\sin D\cos(\alpha - a) \\ &- \sin\delta\cos D\cos(\mathcal{A} - a) + G\,\{r\sin\varphi'\cos d - r\cos\varphi'\sin d\cos(\mu - a)\}\ldots\ldots(15.\,a.) \end{aligned}$$

Æquatio (XV.) vero, per $\cos\delta\cos D\sin d$ multiplicata, efficit

$$\begin{aligned} 0 &= \sin\delta\sin d\cos D\sin(\mathcal{A} - a) - \cos\delta\sin d\sin D\sin(\alpha - a) \\ &+ \frac{\sin d^2}{\cos d}\,.\cos\delta\cos D\sin(\alpha - \mathcal{A})\ldots\ldots(15.\,b.) \end{aligned}$$

Porro, si (15.) et (15. b.) adduntur, erit:

$$\begin{aligned} &(a\,b' - a'\,b)\cos d + (a\,c' - a'\,c)\sin d\cos a - (b\,c' - b'\,c)\sin d\sin a = \\ &\frac{\cos D}{\cos d}\,.\cos\delta\sin(\alpha - \mathcal{A}) - G\,.\,r\cos\varphi'\sin(\mu - a), \end{aligned}$$

unde æquatio (13. d.):

$$\begin{aligned} &(a\,b' - a'\,b)^2 + (a\,c' - a'\,c)^2 + (b\,c' - b'\,c)^2 = \left\{\frac{\cos D}{\cos d}\,.\cos\delta\sin(\alpha - \mathcal{A}) - G\,.\,r\cos\varphi'\sin(\mu - a)\right\}^2 \\ &+ \left[\cos\delta\sin D\cos(\alpha - a) - \sin\delta\cos D\cos(\mathcal{A} - a) + G\,\{r\sin\varphi'\cos d - r\cos\varphi'\sin d\cos(\mu - a)\}\right]^2\ldots(15.\,c.) \end{aligned}$$

Jam vero etiam erat secundum expressionem (II.)

$$\begin{aligned} &(a\,b' - a'\,b)^2 + (a\,c' - a'\,c)^2 + (b\,c' - b'\,c)^2 = (a'\sin\rho \pm a\sin R)^2 \\ &+ (b'\sin\rho \pm b\sin R)^2 + (c'\sin\rho \pm c\sin R)^2, \end{aligned}$$

igitur etiam

$$\begin{aligned} &(a'\sin\rho \pm a\sin R)^2 + (b'\sin\rho \pm b\sin R)^2 + (c'\sin\rho \pm c\sin R)^2 = \\ &\left\{\frac{\cos D}{\cos d}\,.\cos\delta\sin(\alpha - \mathcal{A}) - G\,.\,r\cos\varphi'\sin(\mu - a)\right\}^2 + \\ &\left[\sin\delta\cos D\cos(\alpha - \mathcal{A}) - \cos\delta\sin D\cos(\alpha - a) - G\{r\sin\varphi'\cos d - r\cos\varphi'\sin d\cos(\mu - a)\}\right]^2\ldots(15.\,d.) \end{aligned}$$

Quæ quidem expressio brevitatis causa hanc accipiat formam

$$W^2 = X^2 + Y^2.$$

Tum primum est

$$W^2 = (a'^2 + b'^2 + c'^2) \sin \rho^2 \pm 2\,(aa' + bb' + cc') \sin \rho \sin R + (a^2 + b^2 + c^2) \sin R^2.$$

Deinde pro $a'^2 + b'^2 + c'^2$, $aa' + bb' + cc'$, $a^2 + b^2 + c^2$ earum valores ex (2. a.) et (2. c.) substituuntur, existit

$$W^2 = \Delta'^2 \sin \rho^2 \pm 2\,\Delta\,\Delta' \cos \Sigma \sin \rho \sin R + \Delta^2 \sin R^2;$$

sed

$$\sin \rho = \Delta \sin \rho', \; \sin R = \Delta' \sin R', \cos \Sigma \cos (\rho' \pm R'),$$

igitur

$$\begin{aligned}
W^2 &= \Delta^2 \Delta'^2 \{\sin \rho'^2 \pm 2 \cos (\rho' \pm R') \sin \rho' \sin R' + \sin R'^2\} \\
&= \Delta^2 \Delta'^2 \{(\sin \rho'^2 - \sin \rho'^2 \sin R'^2) + (\sin R'^2 - \sin R'^2 \sin \rho'^2) \pm 2 \sin \rho' \cos \rho' \sin R' \cos R'\} \\
&= \Delta^2\,\Delta'^2 (\sin \rho'^2 \cos R'^2 \pm 2 \sin \rho' \cos R' \cos \rho' \sin R' + \cos \rho'^2 \sin R'^2) \\
&= \Delta^2\,\Delta'^2 \sin (\rho \pm R')^2 \\
&= \Delta^2\,\Delta'^2 \sin \Sigma^2, \text{ et exinde } \Delta^2\,\Delta'^2 \sin \Sigma^2 = X^2 + Y^2.
\end{aligned}$$

Jam habetur

$$\begin{aligned}
\Delta\,\Delta' \sin \Sigma &= (\Delta \sin \rho')\,(\Delta' \cos R') \pm (\Delta' \sin R')\,(\Delta \cos \rho') \\
&= \sin \rho \sqrt{\Delta'^2 - \sin R^2} \pm \sin R \sqrt{\Delta^2 - \sin \rho^2};
\end{aligned}$$

sed secundum (2. c.):

$$\begin{aligned}
\Delta^2 &= a^2 + b^2 + c^2 = 1 - 2\,r \sin \pi \{\cos \delta \cos \phi' \cos (\alpha - \mu) + \sin \delta \sin \phi'\} + r^2 \sin \pi^2 \\
\Delta'^2 &= a'^2 + b'^2 + c'^2 = 1 - 2\,r \sin \pi' \{\cos D \cos \phi' \cos (A - \mu) + \sin D \sin \phi'\} + r^2 \sin \pi'^2.
\end{aligned}$$

Positis igitur hisce expressionibus

$$\left.\begin{aligned}
\sin \phi' \sin \delta + \cos \phi' \cos \delta \cos (\alpha - \mu) &= \cos \gamma \\
\sin \phi' \sin D + \cos \phi' \cos D \cos (A - \mu) &= \cos \gamma'
\end{aligned}\right\} \ldots . (15.\ e.)$$

erit

$$\begin{aligned}
\Delta^2 &= 1 - 2\,r \sin \pi \cos \gamma + r^2 \sin \pi^2 \\
\Delta'^2 &= 1 - 2\,r \sin \pi' \cos \gamma' + r^2 \sin \pi'^2
\end{aligned}$$

et exinde

$$\begin{aligned}
\Delta'^2 - \sin R^2 &= \cos R^2 - 2\,r \sin \pi' \cos \gamma' + r^2 \sin \pi'^2 \\
\Delta^2 - \sin \rho^2 &= \cos \rho^2 - 2\,r \sin \pi \cos \gamma + r^2 \sin \pi^2;
\end{aligned}$$

itaque, si brevitatis causa ponitur:

$$\left.\begin{aligned}
\sqrt{(\cos \rho^2 - 2\,r \sin \pi \cos \gamma + r^2 \sin \pi^2)} &= \lambda \\
\sqrt{(\cos R^2 - 2\,r \sin \pi' \cos \gamma' + r^2 \sin \pi'^2)} &= \lambda'
\end{aligned}\right\} \ldots . (15.\ f.)$$

erit

$$\sqrt{\Delta'^2 - \sin R^2} = \lambda'$$

$$\sqrt{\Delta^2 - \sin \rho^2} = \lambda;$$

ergo

$$\Delta\,\Delta' \sin \Sigma = \lambda' \sin \rho \pm \lambda \sin R,$$

ergo

$$W^2 = (\lambda' \sin \rho \pm \lambda \sin R)^2,$$

et exinde, quia $W^2 = X^2 + Y^2$, si scilicet insuper etiam unum quodque membrum per G diviseris, quæsitam transformationem expressionis (II.) habebis:

$$\left(\frac{\lambda' \sin \rho \pm \lambda \sin R}{G}\right)^2 = \left\{\frac{\cos D}{\cos d}\cdot\frac{\cos \delta \sin(\alpha - A)}{G} - r \cos \varphi' \sin(\mu - a)\right\}^2$$
$$+ \left\{\frac{\sin \delta \cos D \cos(A - a) - \cos \delta \sin D \cos(\alpha - a)}{G} - r\left[\sin \varphi' \cos d - \cos \varphi' \sin d \cos(\mu - a)\right]\right\}^2 \ldots \text{(XVI.)}$$

Expressio (XVI.) formam accepit $k^2 = (P - u)^2 + (Q - v)^2$, eandem, quæ occurrit in fixarum occultationibus, ita tamen, ut k^2 sit quantitas variabilis.

§ 20. Si utraque æquatio in (XIV), secunda scilicet et tertia, per se ipsam multiplicantur, deinde adduntur, hæc existit summa

$$G^2 . \cos d^2 = \sin \pi^2 \cos D^2 - 2 \sin \pi \sin \pi' \cos D \cos \delta \cos(A - \alpha) + \sin \pi'^2 \cos \delta^2,$$

ad quam si prior in (XIV), etiam per se ipsam multiplicata, additur, fit:

$$G^2 = \sin \pi^2 - 2 \sin \pi \sin \pi' \left\{\cos \delta \cos D \cos(A - \alpha) + \sin \delta \sin D\right\} + \sin \pi'^2.$$

Sed factor cum $2 \sin \pi \sin \pi'$ conjunctus est analogus factori cum $\Delta \Delta'$ in (2.) conjuncto, ita tamen, ut quantitatibus quæ insunt in æquatione G^2 definiente non sint adscriptæ lineolæ; est igitur cosinus distantiæ veræ σ centrorum utriusque corporis celestis, quare etiam

$$G^2 = \sin \pi^2 - 2 \sin \pi \sin \pi' \cos \sigma + \sin \pi'^2,$$

ut apud Besselium legitur.

Si porro secunda et tertia æquatio systematis (XIV.) per $\sin A$ et $\cos A$ multiplicantur, et deinde subtrahuntur, existit:

$$G \cos d \sin(A - a) = \sin \pi' \cos \delta \sin(\alpha - A),$$

unde

$$\sin(A - a) = \frac{\sin \pi' \cos \delta \sin(\alpha - A)}{G \cos d},$$

igitur:

$$\tan(A - a) = \frac{\sin \pi' \cos \delta \sin(\alpha - A)}{G \cos \delta \cos(A - a)}:$$

Ex tertia vero æquatione systematis (XIV.) sequitur

$$G . \cos d = \frac{\sin \pi \cos D \sin A - \sin \pi' \cos \delta \sin \alpha}{\sin a};$$

igitur, valore $G . \cos d$ substituto,

$$\tan(A - a) = \frac{\sin \pi \cos \delta \sin(\alpha - A) \sin \alpha}{(\sin \pi \cos D \sin A - \sin \pi' \cos \delta \sin \alpha) \cos(A - a)}.$$

Si hoc loco $\cos(A - a)$ plene scribitur, deinde multiplicatio dicta fit, porro numeratores et nominatores per $\sin a$ dividuntur atque pro $\frac{\cos a}{\sin a}$ ejus valor ex (14. b.) substituitur, hæc denique existit expressio

$$\tan(A - a) - \frac{\sin \pi' \cos \delta \sin(\alpha - A)}{\sin \pi \cos D - \sin \pi' \cos \delta \cos(\alpha - A)} \ldots . (16.)$$

at apud Besselium.

§ 21. Quum in solis eclipsibus quantitas σ semper sit parva, pro æquatione

$$G^2 = \sin \pi^2 - 2 \sin \pi \sin \pi' \cos \sigma + \sin \pi'^2,$$

in antecedenti §. obvia, poni etiam potest:

$$G^2 = \sin \pi^2 - 2 \sin \pi \sin \pi' + \sin \pi'^2,$$

id est

$$G = \sin \pi - \sin \pi'. \ldots (16. \text{a.})$$

Porro pro (16.) poni etiam potest

$$\mathcal{A} - a = \frac{(\alpha - \mathcal{A}) \sin \pi'}{\sin \pi - \sin \pi'},$$

igitur

$$a = \mathcal{A} - \frac{\sin \pi'}{\sin \pi - \sin \pi'} (\alpha - \mathcal{A}),$$

seu tantum

$$a = \mathcal{A} - \frac{\sin \pi'}{\sin \pi} (\alpha - \mathcal{A}) \ldots (16. \text{b.})$$

Denique pro (14. a.) hoc solum poni potest:

$$d = \frac{D \sin \pi - \delta \sin \pi'}{\sin \pi - \sin \pi'} = \frac{D (\sin \pi - \sin \pi') - (\delta - D) \sin \pi'}{\sin \pi - \sin \pi'},$$

id est

$$d = D - \frac{\sin \pi'}{\sin \pi - \sin \pi'} (\delta - D)$$

seu tantum

$$d = D - \frac{\sin \pi'}{\sin \pi} (\delta - D) \ldots (16. \text{c.})$$

PARS SECUNDA.

De usu æquationum expressionumque a Besselio inventarum.

Quum Besselius, quemadmodum sub finem introductionis atque § 7. commemoratum a nobis est, fere nihil monuerit, quomodo formulis inventis (P. I.) in calculo numerico uti possis; tentabimus, secundum formulas partis primæ docere, qua ratione ex observatis occultationibus et solis eclipsibus differentiæ longitudinum numerice derivari possint.

SECTIO PRIMA.

Quomodo ex observata fixæ occultatione longitudines geographicæ sint deducendæ.

§ 1. Besselius in septima et octava dissertationis sectione tria numericarum definitionum genera esse monuit, eorumque lineamenta descripsit. Quorum quidem generum primum ceteris duobus omni ex parte præferendum esse contendenti, omnibus tribus examinatis, jure meritoque assentior, quare non nisi in illo primo genere describendo acquiescemus.

§ 2. Si plures observationes unius occultationis fixæ calculandæ sunt, primum quodvis observationis tempus per media tempora solaria et sideralia exprimatur.* Deinde tempora media solaria ope longitudinum geographicarum, quamvisnondum satis accurate definitarum, in media tempora Berolinensia reducantur, atque ex eorum summa medium T derivetur, quod tantum ad quadrantes horæ usque exprimatur necesse est. Porro tempora sideralia per gradus expressa efficiunt μ. Postremo ex ephemeridibus Berolinensibus, ab Enckio editis, petantur argumenta α, δ et π, auxilio interpolationis formula (VII.) quam in hanc formam redegimus:

$$y = a + X.\, b + X'.\, c + X''.\, d + X'''.\, e + \ldots. \quad (1.),$$

ubi coëfficientes X, X', X'', X''', etc. ex tabula, quæ commentationi nostræ adjecta est, et in qua coëfficientes ad singulos horæ quadrantis pertinentes computati sunt, peti possunt. Commodius est, argumenta α, δ et π rectascensionem, declinationem et parallaxin lunæ; A et D vero fixæ rectascensionem et declinationem ab aberratione et nutatione affectam habere.

§ 3. Ex elevationibus polinotis ϕ observatoriorum secundum (7. a.) definiantur quantitates w et w' per formulas

$$w = \frac{\cos\phi}{\sqrt{1 - e^2 \sin\phi^2}} \quad (2.) \qquad w' = \frac{(1 - e^2)\sin\phi}{\sqrt{1 - e^2 \sin\phi^2}} \ldots. \quad (3.)$$

Nota.—Si Telluris compressio $\chi = \frac{1}{302}$ sumitur, est $\log e^2 = 7.8203066$, $\log(1 - e^2) = 9.9971191$.

Porro pro tribus temporibus $T - 1^h$, T^h, $T + 1^h$, quantitates P et Q calculentur, secundum (4.) partis primæ, per expressiones:

$$P = \frac{\cos\delta \sin(\alpha - A)}{\sin\pi} \ldots. \quad (4.)$$
$$C = \sin\delta \cos D \ldots. \quad (5.)$$
$$C' = \cos\delta \sin D \cos(\alpha - A) \ldots. \quad (6.)$$
$$Q = \frac{C - C'}{\sin\pi} \ldots. \quad (7.)$$

atque deinde, quod attinet ad utramque quantitatem P et Q formetur hoc schema:

$$\left.\begin{array}{r|lll} T - 1^h & a_, & & \\ T^h & a & \begin{array}{l} b_, \\ b' \end{array} & c \qquad \dfrac{b_, + b'}{2} = b \\ T + 1^h & a' & & \end{array}\right\} \ldots. (8.),$$

quod vero attinet ad utramque quantitatem p' et q', hoc:

$$\begin{array}{r|l} T - 1^h & b - \frac{1}{2}c \\ T^h & b \qquad \ldots. (9.). \\ T + 1^h & b + \frac{1}{2}c \end{array}$$

* Besselius commodius quidem esse censet (cf. dissert. sect. 6. pag. 129) temporibus observationum, per media, vera seu etiam sideralia tempora expressis ipsis in calculo uti, deinde formulam (6.) per numerum secundarum ejus temporum generis multiplicare, quod horam in calculo sumtam æquiparat, ut hac ratione differentiam meridianorum per secundas horæ expressam invenias; sed nemo non intelligit, si plura observationum tempora per diversa temporum genera expressa sint, etiam numerum secundarum, cui Besselius omnino signum s indidit, non semper eundem esse posse, ideoque, si respiciatur valor variabilis s, vitia comitti posse, qua propter computationes ipsas irritas fieri necesse sit.

Denique definiantur quantitates T' per æquationem $T'=t-T-d$, quæ quidem quantitates per horas earumque partes decimales expressæ inveniuntur, atque valores quantitatum, p' et q', quæ attinent ad tempora $T+T'$ ope schematis (9.) per interpolationem quærantur.

§ 4. Nunc demum calculus ipse incipit, qui per se spectatus summa est simplicitate et facilitate. Nam secundum (4.) (5. b.) et (5. c.) quantitates M, $\log m$, N, $\log n$ et ψ invenies per æquationes

$$u = w \sin(\mu - A) \quad \ldots\ldots (8^*.)$$
$$v = w' \cos D - w \sin D \cos(\mu - A) \quad \ldots\ldots (9^*.)$$
$$m \sin M = p - u \quad \ldots\ldots (10.)$$
$$m \cos M = q - v \quad \ldots\ldots (11.)$$
$$n \sin N = p' \quad \ldots\ldots (12.)$$
$$n \cos N = q' \quad \ldots\ldots (13.)$$
$$\cos \psi = \frac{m \sin(M-N)}{k} \quad \ldots\ldots (14.)$$
$$h = \frac{s}{n\,\omega \sin \pi} \quad \ldots\ldots (15.),$$

ubi p et q significant valores quantitatum P et Q tempore T, ideoque quovis observatorio usurpari possunt. Hoc vero tenendum est, angulum ψ intra 0° et 180° sumendum esse, si observationes sint immersiones, si vero emersiones sint, intra 180° et 360°. Ultimo loco definiantur quantitates T'', quæ etiam per minutas horæ expressæ inveniuntur, per æquationem

$$T'' = \frac{m\,s \cos(M-N-\psi)}{n \cos \psi} \quad \ldots\ldots (16.),$$

ubi $s = 60'$, $\log s = 1.7781512$, et in (14.) $\log k = 9.4353665$. Jam si valorem $t-T$, non, ut supra, per horas earumque partes decimales, sed per horæ minutas earumque partes decimales expresseris, derivabuntur secundum (XII.) veræ d ex formula

$$d = t - T + T'' + h\,.\,\varepsilon + h \cot \psi \zeta \quad \ldots\ldots (17.)$$

et habebis ad quantitates $\Delta\alpha$, $\Delta\delta$ definiendas secundum (11.) primæ partis æquationes:

$$+\sin N \cos\delta\,.\,\Delta\alpha + \cos N\,.\,\Delta\delta = \varepsilon \quad \ldots\ldots (18.)$$
$$-\cos N \cos\delta\,.\,\Delta\alpha + \sin N\,.\,\Delta\delta = \zeta \quad \ldots\ldots (19.)$$

Nota.—Minus aptum certe mihi videtur esse, æquationibus (17.), (18.) et (19.) uti, quam expressione sequente secundum (XI.):

$$d = t - T + T'' - T'''\,.\,\Delta\alpha + T^{\mathrm{iv}}\,.\,\Delta\delta \quad \ldots\ldots (17.^*)$$

ubi

$$T''' = \frac{h \cos(N+\psi)\cos\delta}{\sin\psi} \quad \ldots\ldots (18.^*)$$

et

$$T^{\mathrm{iv}} = \frac{h \sin(N+\psi)}{\sin\psi} \quad \ldots\ldots (19.^*)$$

SECTIO SECUNDA.

Quomodo ex observata solis eclipsi longitudines geographicæ sint deducendæ.

§ 5. Besselius in ultima dissertationis sectione solis eclipses fere secundum easdem regulas quam fixarum occultationes computandas esse docuit.

§ 6. Primum aggrediaris rem secundum eandem rationem, quam § 2. descripsimus. Deinde secundum (7. a.) definiantur quantitates r cos φ, r sin φ' per formulas

$$r \cos \varphi' = \frac{\cos \varphi}{\sqrt{1 - e^2 \sin \varphi^2}} \quad \ldots . (2.^*)$$

$$r \sin \varphi' = \frac{(1 - e^2) \sin \varphi}{\sqrt{1 - e^2 \sin \varphi^2}} \quad \ldots . (3.^*);$$

porro habebis pro æquationibus (4.), (5.), (6.) et (7.) § 3. hujus partes expressiones P et Q definientes, secundum (XVI.)

$$P = \frac{\cos D \cos \delta \sin (\alpha - A)}{G \cos d} \quad \ldots . (4.^*)$$

$$Q = \frac{\sin \delta \cos D \cos (A - a) - \cos \delta \sin D \cos (\alpha - a)}{G} \quad \ldots . (5.^*)$$

ubi G, a et d {secundum (16. a.), (16. b.) et (16. c.) § 21.} per æquationes

$$G = \sin \pi - \sin \pi' \quad \ldots . (20.)$$

$$a = A - \frac{\sin \pi'}{\sin \pi} (\alpha - A) \quad \ldots . (21.)$$

$$d = D - \frac{\sin \pi'}{\sin \pi} (\delta - D) \quad \ldots . (22.)$$

definiuntur. In plurimis solis eclipsibus formulæ approximatæ plane satisfacient:

$$P = \frac{\cos \delta \sin (\alpha - A)}{G} \quad \ldots . (4.^{**})$$

$$Q = \frac{\sin (\delta - D)}{G} \quad \ldots . (5.^{**})$$

quoniam valores quantitatum G, a et d dati non nisi approximati sunt.

§ 7. Quod attinet ad formulas (8.) usque ad (15.) § 3. et § 4. hujus partis (excepta quantitate k), secundum quas fixarum occultationes sunt computandæ, etiam in solis eclipsibus valent. Sed valores u et v non secundum (8*.) et (9*.), sed secundum (XVI) per æquationes

$$u = r \cos \varphi' \sin (\mu - a) \quad \ldots . (23.)$$

$$x = r \sin \varphi' \cos d \quad \ldots . (24.)$$

$$y = r \cos \varphi' \sin d \cos (\mu - a) \quad \ldots . (25.)$$

$$v = x - y \quad \ldots . (26.)$$

definiendi sunt. Quantitas k, si solis eclipses fiunt, non constans, sed variabilis est, quemadmodum sub finem § 19. primæ partis monuimus, ea de causa, quod pendet ab observatorio et a complementibus altitudinum solis atque lunæ. Itaque secundum (15. e.) et

(15. f.) § 19. hac ratione definienda est. Primum computentur quantitates trigonometricæ $\cos\gamma$ et $\cos\gamma'$, per expressiones

$$\cos\gamma = \sin\phi' \sin\delta + \cos\phi' \cos\delta \cos(\mu - a) \;\dots\; (27.)$$
$$\cos\gamma' = \sin\phi' \sin D + \cos\phi' \cos D \cos(\mu - A) \;\dots\; (28.),$$

porro

$$\lambda = \sqrt{\cos\rho^2 - 2\, r \sin\pi \cos\gamma + \sin\pi^2} \;\dots\; (29.)$$
$$\lambda' = \sqrt{\cos R^2 - 2\, r \sin\pi' \cos\gamma' + \sin\pi'^2} \;\dots\; (30.),$$

denique

$$k = \frac{\lambda' \sin\rho \pm \lambda \sin R}{G} \;\dots\; (31.)$$

idque nonnisi positive.

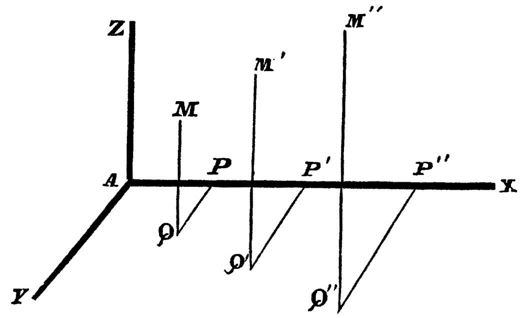
Quum vero $\cos\varrho^2$ et $\cos R^2$ prope $= 1$, atque $r^2 \sin\pi^2$ et $r^2 \sin\pi'^2$ quippe quæ fere semper sunt minimæ quantitates, salva calculi integritate negligi possint; pro (29.) et (30.) satisfacient expressiones approximatæ:

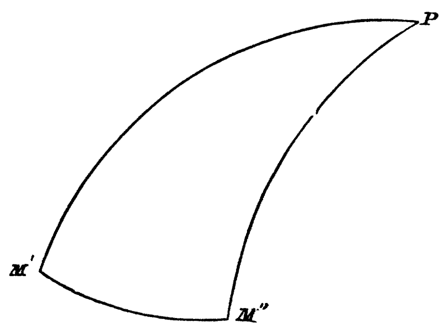
$$\lambda = \sqrt{1 - 2\, r \sin\pi \cos\gamma} \;\dots\; (29.^*)$$
$$\lambda' = \sqrt{1 - 2\, r \sin\pi' \cos\gamma'} \;\dots\; (30.^*)$$

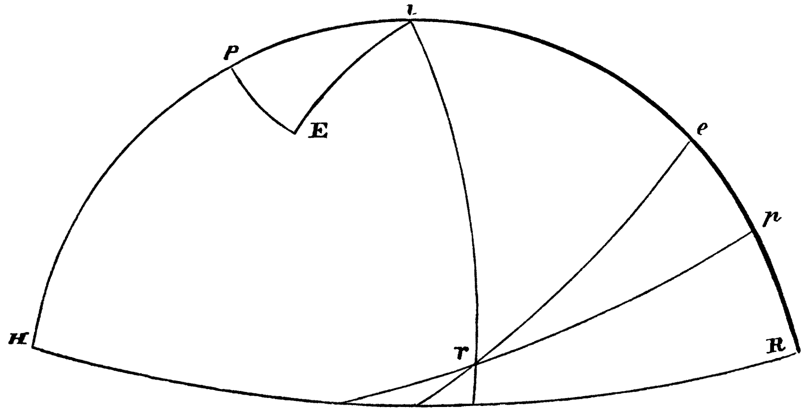
Calculus residuus, excepta quantitate k, eodem modo secundum formulas (16.) usque ad (19*.) fit, quem § 4, demonstravimus.

TABULA INTERPOLATIONIS.

$\pm x$	$\pm X$	$+ X'$	$\mp X''$	$- X'''$
0^h	0.0000	0.0000	0.0000	0.000
1	0.0833	0.0035	0.0138	0.000
2	0.1667	0.0139	0.0270	0.001
3	0.2500	0.0313	0.0391	0.002
4	0.3333	0.0556	0.0494	0.004
5	0.4167	0.0868	0.0574	0.006
6	0.5000	0.1250	0.0625	0.008


Z
M''
M'
M
A
P
P'
P''
X
Q
Q'
Q''
Y


P
M'
M''


P
E
e
p
H
r
R